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M. B. BARBAN, A. I. VINOGRADOV

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Abstract

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M. B. BARBAN, A. I. VINOGRADOV

ON THE NUMBER-THEORETIC BASIS OF PROBABILISTIC NUMBER THEORY

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The number-theoretic basis of probabilistic number theory is provided by fundamental lemmas 1 and 2 (see (1)). The estimate of the remainder term in the central limit theorem (c.l.t.) for additive arithmetic functions depends on the degree of accuracy of these lemmas.

Let $\alpha = \ln N / \ln r$. It is well known that, by Brun' s or Selberg' s sieve method, for the number of integers not exceeding N and not divisible by primes not exceeding r, one obtains an asymptotic formula with a decrease of the form $\exp\{-c_1 \alpha \ln \alpha\}$, where $c_1 = \text{const}$ (see, for example, (2)). This theorem constitutes the content of fundamental lemma 1. Further, let $\beta = \ln u / \ln r$.

If we denote by

$$f_r(n) = \prod_{\substack{p^\alpha | n \\ p<r}} p^\alpha$$

and by $M^u_{N,r}$ the number of integers not exceeding N and satisfying the condition $f_r(n) \geq u$, then fundamental lemma 2 consists in estimating $M^u_{N,r}$. At first in probabilistic number theory the easily attainable estimate

$$M^u_{N,r} < c_2 \frac{N}{\beta}$$

was used.

In (3) this estimate was improved by means of a very elementary "method of moments" to

$$M^u_{N,r} < N \exp\{-c_3 \beta\},$$

which made it possible to strengthen the remainder term in the above-mentioned c.l.t. Then R. V. Uzhdavinis (4) showed that as soon as, for $M^u_{N,r}$, a decrease exponential in β is achieved, one can immediately use, for the derivation of the c.l.t., a simpler (and stronger) form of the law of large numbers. This yielded an even better remainder term in the c.l.t.

In the present note we shall show that the estimate

$$M^u_{N,r} < N \exp\{-c_4 \beta \ln \beta\} \tag{1}$$

is an easy consequence of the estimates of (5), and from this we shall derive a new strengthening of the c.l.t. (apparently the limiting one for probabilistic methods in their modern form).

We begin with estimate (1). Every n is uniquely representable in the form $n = f_r(n)m$. The sum

$$M_{N,r}^{u} = \sum_{\substack{n<N \\ f_r(n)\geq u}} 1$$

we split into two parts, corresponding to the cases $m = 1$ and $m > 1$. For $m = 1$ we simply have numbers with small prime divisors. We use the estimates of (5), which, evidently, may be rewritten in the following roughened form

$$\sum_{\substack{n\leq N \\ (n,p)=1,\ p>r}} 1 < N\exp\{-c_5\alpha\ln\alpha\} \tag{2}$$

under the natural (and necessary) restriction on α,

$$\alpha < \ln N/\ln\ln N.$$

Thus,

$$M_{N,r}^{u} < N\exp\{-c_5\alpha\ln\alpha\} + \sum_{u\leq d<N/r}\ \sum_{m\leq N/d} 1, \tag{3}$$

where d runs through numbers consisting only of primes not exceeding r, and m runs through numbers not divisible by primes smaller than r. The known estimates of the sieve method $(x > r)$

$$\sum_{m\leq x} 1 < c_6\frac{x}{\ln r}$$

allow one to estimate the sum in (3) by the quantity

$$c_6\frac{N}{\ln r}\sum_{d>u}\frac{1}{d}. \tag{4}$$

The latter is easily estimated with the aid of (2):

$$\sum_{d>u}\frac{1}{d} \leq \sum_{k=0}^{\infty}\ \sum_{2^k u<d\leq 2^{k+1}u}\frac{1}{d} <$$

$$< \exp\{-c_5\beta\ln\beta\}\sum_{k=0}^{\infty}\exp\left\{-c_7\frac{k\ln\beta}{\ln r}\right\} < c_8\frac{\ln r}{\ln\beta}\exp\{-c_5\beta\ln\beta\}, \tag{5}$$

where again the natural restriction arises $\beta < \ln u/\ln\ln u$. Substitution of (5) and (4) into (3) completes the proof of (1).

With the aid of estimate (1) the following limit theorem is proved.

Theorem. Let $f(m) = \sum_{p|m} f(p)$ be a strongly additive arithmetical function satisfying the condition $\Lambda_n/B_n \le \mu_n$, where μ_n is nonincreasing and tends to 0 as $n \to \infty$,

$$\Lambda_n = \max_{p\le n} |f(p)|, \qquad B_n^2 = \sum_{p\le n} \frac{f^2(p)}{p},$$

$$A_n = \sum_{p\le n} \frac{f(p)}{p}, \qquad G(x) \text{ is the normal law.}$$

Then, uniformly with respect to x, one has

$$\frac{1}{n} N \left\{ \frac{f(m) - A_n}{B_n} < x,\ m \le n \right\} = G(x) + O\left(\frac{\mu_n \ln 1/\mu_n}{\ln\ln 1/\mu_n} \right).$$

The proof proceeds analogously to (1), except that as r for "truncating" $f(m)$ one should choose

$$x = n^{c_9} \frac{\ln\ln 1/\mu_n}{\ln 1/\mu_n},$$

and, in estimating high moments for the law of large numbers, use the considerations of R. V. Uzhdavinis:

$$\sum_{m\le n} \left\{ f(m) - \sum_{\substack{p|m \\ p\le r}} f(p) \right\}^l \le n\mu_n^l B_n^l \max_{m\le n} \left\{ \sum_{\substack{p|m \\ p>r}} 1 \right\}^l \le n\mu_n^l B_n^l \left\{ \frac{\ln n}{\ln r} \right\}^l,$$

since every number not exceeding n has no more than $\ln n/\ln r$ prime divisors greater than r.

Institute of Mathematics named after V. I. Romanovskii
Academy of Sciences of the Uzbek SSR

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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