MATHEMATICS

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Abstract

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ON THE DISTRIBUTION OF PRIME NUMBERS IN SHORT PROGRESSIONS mod p^n

It is known that the study of the problem of the distribution of prime numbers in arithmetic progressions requires considerable information about the "critical" zeros of L-functions. For example, obtaining an asymptotic law for primes in the progression

$$q \equiv l(\operatorname{mod} D), \qquad (l, D) = 1, \qquad q \ll D^{2+\varepsilon}, \quad D \to \infty,$$

requires the validity of the extended Riemann hypothesis.* Therefore the investigation of the distribution of primes in progressions of general type encounters difficulties. However, there is every reason to believe that for partial sets of values of D the problem will prove easier. In this note we show that for $D = p^n$ (p a fixed prime, $n = 1, 2, ...$) one can obtain estimates close to hypothetical ones. The following is proved.

Theorem. Let $D = p^n$, $x \gg D^{8/3+\varepsilon}$. Then

$$\pi(x, D, l) = h^{-1} \operatorname{li} x + O(xh^{-1}\eta), \tag{1}$$

where $h = \varphi(D)$, $\eta = O((\lg D)^{-A})$, A is any positive number.

Moduli of the indicated type have already been considered in the literature (see ([1])). For them it naturally turns out to be useful to employ the theory of functions of a p-adic variable.

The proof of (1) is based on several lemmas. Let

$$\psi_0(x) = \psi(x, D, l) = h^{-1} \sum_{\chi} \overline{\chi}(l) \sum_{n \le x} \Lambda(n)\chi(n),$$

$$\psi_m(x) = \int_0^x \psi_{m-1}(x)\, dx \qquad (m = 0, 1, 2, ...),$$

$$\psi_m(x) = x^{m+1} h^{-1} \left(\frac{1}{(m+1)!} + \delta_m(x) \right).$$

The classical theory shows that the derivation of (1) reduces to obtaining analogous estimates for $\delta_0(x)$; moreover, one can show that

$$\delta_0(x) \ll \delta_m^{\frac{1}{2m+1}}, \qquad x > x_0(D),$$

where $\delta_m = \sup|\delta_m(x)|$ for $x_0 \le x < \infty$, if there exists such an x_1 that for $x > x_1$ the estimate $\delta_m \ll (\lg D)^{-A}$ is valid. We shall show that $x_1 = D^{8/3+\varepsilon}$.

Lemma 1.

$$\delta_m(x) = -\sum_{\chi} \overline{\chi}(l) \sum_{\rho(x)} \frac{x^{\rho-1}}{\rho(\rho+1)\cdots(\rho+m)} + O(x^{-1} h \lg^2 x),$$

where $\rho(x)$ is the aggregate of critical zeros of $L(s,\chi)$, $m \ge 1$.

This lemma for $m = 1$ is an analogue of the well-known identity for $\psi_1(x)$ in the case $D = 1$ (see, for example, ([2]), p. 43). Termwise m-fold integration and shifting the contour to the line $\sigma = -\frac{1}{2}$ prove the lemma.

* Or a very strong density hypothesis.

Lemma 2. If, for some set $\{D\}$, the following facts hold:

1) $N(\sigma, T) \ll T^A D^{B(1-\sigma)} \lg^c D$, where $N(\sigma,T)$ is the total number of zeros of all $L(s,\chi)$ mod D in the rectangle $\sigma \le \sigma' \le 1$; $|t| \le T$;
2) $L(s,\chi)$ in the region

$$\sigma \ge \sigma_0 = 1 - c_1 (\lg D)^{-\alpha}, \quad \alpha < 1, \quad |t| \le \tau = (\lg D)^2,$$

then, for $x \ge D^{B+\varepsilon}$, $\varepsilon > 0$, the estimate

$$\delta_m(x) \ll \tau^{-1}$$

holds.

For the proof we choose m so large that the series $\sum_{\nu=1}^{\infty} \nu^{A-m}$ converges rapidly. Then we select in the identity for $\delta_m(x)$ all those ρ for which $|\operatorname{Im}\rho| \le (\lg D)^L = \tau$. The estimate of the sum over such ρ gives:

$$\sum_{\chi} \bar{\chi}(l) \sum_{\rho}{}' \frac{x^{\rho-1}}{\rho(\rho+1)\ldots(\rho+m)} \ll \int_0^{\sigma_0} x^{\sigma-1} N(\sigma,\tau)\, d\sigma +$$

$$+ O(x, \tau^{-1} D \lg \tau D) \ll \exp\{-c_1 (\lg D)^{1-\alpha}\}, \quad c_1 > 0.$$

We distribute the remaining part of the sum into strips $\nu \le |t| < \nu + 1$ and again apply within these strips the estimate for $N(\sigma,\tau)$. The choice of the number m

is such that the total contribution to $\delta_m(x)$ coming from all strips is of order $O(\tau^{-1})$, which proves the lemma.

Lemma 3. If
$$|L(1/2+it,\chi)| \le M(D)(|t|+2)^{c_0},$$
then
$$N(\sigma,T) \ll T^{1+2c_0}(M^2D)^{2(1-\sigma)}\lg^7 D$$
in the rectangle $\sigma \le \sigma' \le 1,\ |t| \le T$, for any $\sigma \ge 0,\ T \ge 2$.

For the proof of the lemma see ([3]), p. 422.

Lemma 4. If $D = p^n$, then $L(s,\chi) \ne 0$ in the region
$$\sigma \ge 1 - \lg^{0.9} D, \quad |t| \le \lg^L D$$
($L > 0$ may be arbitrary).

The proof of this lemma is based on estimates for sums of values of the characters $\chi(\nu)$ mod D, obtained in the work ([4]).

Lemma 5. If $D = p^n$, then
$$L(1/2+it,\chi) \ll D^{1/6}\lg^2 D\,(|t|+1).$$

This lemma follows directly from the identity
$$L(1/2+it,\chi) = (1/2+it)\int_0^\infty S(x)x^{-3/2-it}\,dx,$$
where
$$S(x) = \sum_{\nu\le x}\chi(\nu),$$
provided one shows that
$$S(x) \ll D^{1/6}\sqrt{x\lg D}$$
in the interval $(D^{1/3}, D^{2/3})$. The estimate in this interval is essentially reduced to the investigation of the sum
$$S = \sum_{\nu=N}^{N'}\chi(\nu),$$
where $p^s \ll N \ll N' < 2N \ll D^{2/3}$, and s is the least natural number $\ge (n+2)/3$. Putting
$$\nu = l + p^s u, \quad (l,p)=1, \quad ll^* \equiv 1 \pmod{p^n},$$
we obtain
$$S \ll p^{s/2}S_1^{1/2},$$

where

$$S_1 = \sum_{(l,p)=1}^{p^s} \left| \sum_{N_1 \leqslant u \leqslant N_2} \chi(1 + l^* u p^s) \right|^2 .$$

But, by a theorem of A. G. Postnikov ((1), p. 217) and by virtue of the choice of the number s:

$$\chi(1 + l^* u p^s) = \exp 2\pi i(\alpha u^2 + \beta u), \qquad \alpha = \Lambda a_2' l^{*2} p^{-r},$$

$$\beta = \Lambda l^* p^{s-n}, \qquad r = n - 2s, \qquad a_2 = a_2' p, \qquad p \nmid a_2',$$

where Λ and a_2 have the same meaning as in (1).

Thus the estimation of S_1 is reduced to the investigation of the sum

$$\left| \sum_{N_1}^{N_2} \exp 2\pi i(\alpha u^2 + \beta u) \right|^2 ,$$

whose estimation is carried out by well-known methods. After simple computations we have:

$$S_1 \ll \sum_{|u-u'| \leqslant N p^{-s}} \sum_{l=0}^{p^s - 1} \min\left(\frac{N}{p^s}, \frac{1}{\{\alpha(u-u')\}} \right) \ll \left(\frac{N^2}{p^{2s}} + N \right) \lg D.$$

Here it must be taken into account that if l runs through a reduced system of residues $\bmod p^s$, then l^{*2} runs through the same system, but with possible repetitions of multiplicity $\ll 2p^\delta$. Since $N^2 p^{-2s} \ll N$ for $N \leqslant D^{2/3}$, we obtain the required estimate of $S(x)$.

Comparing all the lemmas listed above, we obtain the proof of the main theorem.

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Note: Figure translations are in progress. See original paper for figures.