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Abstract

Full Text

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ON THE QUESTION OF THE APPLICABILITY OF A LIMIT THEOREM OF ORDER $p > 0$ TO AN INHOMOGENEOUS MARKOV CHAIN WITH TWO STATES

(Presented by Academician S. N. Bernstein on 23 VII 1963)

An inhomogeneous Markov chain is considered in the form of a sequence of series of random variables

$$x_{k1}, x_{k2}, \dots, x_{kk} \qquad (k = 1, 2, \dots, n),$$

each of which assumes only the values 0 and 1.

The transition matrix is

$$p_{ki} = \begin{vmatrix} p_{11}^{(i)}(k) & p_{10}^{(i)}(k) \\ p_{01}^{(i)}(k) & p_{00}^{(i)}(k) \end{vmatrix}, \qquad i = 1, 2, \dots, k.$$

Here $p_{\alpha\beta}^{(i)}(k)$ denotes the transition probability; the first index α indicates the value of the preceding random variable x_{i-1}, and the second index β the value of the random variable x_i; k is the number of the series (in what follows the index k will be omitted). To specify the Markov chain completely it is necessary to give $\mathbf{P}(x_1 = 1) = p_1$ and $\mathbf{P}(x_1 = 0) = q_1$.

Following S. N. Bernstein ([1]), we shall say that a limit theorem of order $p > 0$ is applicable to the sum $s_n = x_1 + x_2 + \cdots + x_n$ if the following two limiting relations hold simultaneously:

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{s_n - Ms_n}{\sqrt{B_n}} < t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-z^2/2}\, dz,$$

$$\lim_{n\to\infty} \mathbf{M}\left|\frac{s_n - Ms_n}{\sqrt{B_n}}\right|^p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |t|^p e^{-t^2/2}\, dt,$$

where $B_n = \mathbf{M}(s_n - Ms_n)^2$.

In the present note several theorems are formulated in which necessary and sufficient conditions are indicated for the applicability of a limit theorem of arbitrary order $p > 0$ to the sum of quantities connected in an inhomogeneous Markov chain with two states.

For brevity of exposition we introduce a number of notations and definitions.

$$p_i = \mathbf{P}(x_i = 1), \qquad q_i = \mathbf{P}(x_i = 0), \qquad \delta_i = p_{11}^{(i)} - p_{01}^{(i)},$$

$$\Pi_{i+1,i+k} = \delta_{i+1}\delta_{i+2}\cdots\delta_{i+k} \qquad (\Pi_{i+1,i} = 1/2),$$

$$T_i = \sum_{k=0}^{n} \Pi_{i+1,i+k}, \qquad T_i^+ = \sum_{k=0}^{n} \left|\Pi_{i+1,i+k}\right|.$$

It is known that

$$B_n = \sum_{i=1}^{n} p_i q_i T_i, \qquad B_n^+ = \sum_{i=1}^{n} p_i q_i T_i^+.$$

We shall call the sequence $x_j, x_{j+1}, \dots, x_{j+k}$ a Markov chain on the interval $[j, j+k]$. In particular, the sequence $x_1, x_2, \dots, x_n$ will be called a Markov chain on the whole interval. A Markov chain will be called positively stable on the interval $[j, j+k]$ if $\delta_i \geq 0$ for all i satisfying the inequality $j \leq i \leq j+k$. If $\delta_i \leq 0$ for all i satisfying the inequality $j \leq i \leq j+k$, then such a Markov chain will be called negatively st-

stable on the interval $[j, j+k]$. We shall say that a Markov chain decomposes into two independent chains if there exists such an i for which $\delta_i = 0$.

Theorem 1. If

$$\overline{\lim} \frac{B_n^+}{B_n} = C < \infty, \qquad \lim_{n\to\infty} \frac{\max_{1\leq i\leq n-1} T_i^+}{\sqrt{B_n}} = 0, \tag{1}$$

then the limit theorem of arbitrary order $p > 0$ is applicable to the sum s_n.

Theorem 2. If the Markov chain is positively stable on the entire interval and

$$\lim_{n\to\infty} \frac{\max_{1\leq i\leq n-1} T_i}{\sqrt{B_n}} = 0, \tag{2}$$

then the limit theorem of arbitrary order $p > 0$ is applicable to the sum s_n.

Theorem 3. If the Markov chain is negatively stable on the entire interval and condition (1) is satisfied, then the limit theorem of arbitrary order $p > 0$ is applicable to the sum s_n.

Theorem 4. If the Markov chain is positively stable and

1) $0 < p_1 - q_1 < 1$,

2) $p_{01}^{(i)} \geq p_{10}^{(i)}$,

then (2) is a necessary condition for applicability to the sum s_n of the limit theorem of order $p \geq 4$.

Theorem 5. If the Markov chain is negatively stable and

1) $0 < q_1 - p_1 < 1$,

2) $p_{01}^{(i)} \geq p_{10}^{(i)}$,

then (2) is a necessary condition for applicability to the sum s_n of the limit theorem of order $p \geq 4$.

Theorem 6. If the Markov chain decomposes into two independent chains, each of which is either positively stable or negatively stable, then (1) is a sufficient condition for applicability to s_n of the limit theorem of order $p \geq 4$ and a necessary condition if:

1) $B_n \to \infty$ as $n \to \infty$,

2) $p_{01}^{(i)} = p_{10}^{(i)}$,

3) $p_1 = q_1$.

To illustrate these theorems we give two examples.

Example 1. Let $p_{01}^{(n)} \sim \frac{b}{n^\beta}$, $p_{10}^{(n)} \sim \frac{a}{n^\alpha}$, where $a > 0$, $b > 1$, $\alpha \leq \beta$. Then

$$1 - \frac{a+b}{n^\alpha} \leq \delta_n \leq 1 - \frac{a}{n^\alpha}$$

for sufficiently large n. As shown in [2], $B_n \geq ln^{1+2\alpha-\beta}$, where $l > 0$. Therefore

$$\frac{2}{(a+b)n^{1-\alpha}} \leq \frac{1}{\sqrt{B_n}} \max_{1 \leq i \leq n-1} T_i \leq \frac{1}{a\sqrt{l}} \frac{1}{n^{(1-\beta)/2}},$$

since $B_n \leq \frac{1}{4}n^2$. Hence it follows that for $\beta < 1$ the limit theorem of arbitrary order $p > 0$ is applicable, whereas for $\alpha \geq 1$ it is not applicable.

Example 2 (S. N. Bernstein [1]). Let

$$p_{01}^{(i)} = p_{10}^{(i)} = \frac{1}{an^{1/3}}, \qquad i \leq n^{1/3};$$

$$p_{11}^{(1)} = p_{01}^{(i)} = \frac{1}{2}, \qquad i = n^{1/3} + 1; \qquad p_{11}^{(i)} = p_{00}^{(i)} = \frac{1}{n^{1/3}}, \quad i > n^{1/3} + 1.$$

In this case $B_n \sim c^2 n^{2/3}$. Applying Theorem 6, it is not difficult to show that condition (1) is not fulfilled. Therefore the limit theorem is not applicable.

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References

1. S. N. Bernstein, *UMN*, 10, 65 (1944).
2. B. V. Shirokorad, *Izv. AN SSSR, ser. matem.*, 18, No. 1 (1954).

Note: Figure translations are in progress. See original paper for figures.