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Abstract

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EXPANSIONS IN DOUBLE HYPERGEOMETRIC FUNCTIONS OF KAMPÉ DE FÉRIET OF HIGHER ORDERS

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Introduction. J. Kampé de Fériet ([1], [2]) introduced the following double hypergeometric functions of higher order (with many parameters) of two variables:

$$F\left(\begin{array}{c|c|c} \mu & \alpha_1,\dots,\alpha_\mu & \\ \nu & \beta_1,\beta_1';\dots;\beta_\nu,\beta_\nu' & \\ \rho & \gamma_1,\dots,\gamma_\rho & x,y \\ \sigma & \delta_1,\delta_1';\dots;\delta_\sigma,\delta_\sigma' & \end{array}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\alpha_j;m+n)\prod_{j=1}^{\nu}\{(\beta_j;m)(\beta_j';n)\}}{\prod_{j=1}^{\rho}(\gamma_j;m+n)\prod_{j=1}^{\sigma}\{(\delta_j;m)(\delta_j';n)\}}\frac{x^m y^n}{(1;m)(1;n)}, \tag{1}$$

where $\mu+\nu<\rho+\sigma+1$ (for the properties and definitions of these functions see ([3]), pp. 149-176).

For special values of the parameters μ,ν,ρ, and σ, the functions (1) reduce to the four double hypergeometric functions of two variables. Thus we have (([3]), p. 151)

$$F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta,\beta' & \\ 1 & \gamma & x,y \\ 0 & \dots & \end{array}\right) = F^{[1]}[\alpha;\beta,\beta';\gamma;x,y]; \tag{2}$$

$$F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta,\beta' & x,y \\ 0 & \dots & \\ 1 & \delta,\delta' & \end{array}\right) = F^{[2]}[\alpha;\beta,\beta';\delta,\delta',x,y]; \tag{3}$$

$$F\left(\begin{array}{c|c|c} 0 & \dots & \\ 2 & \beta_1,\beta_1';\beta_2,\beta_2' & \\ 1 & \gamma & x,y \\ 0 & \dots & \end{array}\right) = F^{[3]}[\beta_1,\beta_1';\beta_2,\beta_2';\gamma;x,y]; \tag{4}$$

$$F\left(\begin{array}{c|c|c} 2 & \alpha_1, \alpha_2 & \\ 0 & \dots & \\ 0 & \dots & x, y \\ 1 & \delta_1, \delta_1' & \end{array}\right) = F^{[4]}[\alpha_1, \alpha_2; \delta_1, \delta_1'; x, y], \tag{5}$$

where $F^{[1]}, F^{[2]}, F^{[3]}$, and $F^{[4]}$ are Appell functions (([3]), p. 14).

It is easy to see that

$$F\left(\begin{array}{c|c|c} \mu & \alpha_1, \dots, \alpha_\mu & \\ 0 & \dots & \\ \rho & \gamma_1, \dots, \gamma_\rho & x, y \\ 0 & \dots & \end{array}\right) = F\left(\begin{array}{c} \alpha_1, \dots, \alpha_\mu; \quad x + y \\ \gamma_1, \dots, \gamma_\rho \end{array}\right); \tag{6}$$

$$F\left(\begin{array}{c|c|c} 0 & \cdot \qquad \cdot \;\; \cdot \;\; \cdot & \\ \nu & \beta_1, \ \beta_1'; \dots; \beta_\nu, \ \beta_\nu' & \\ 0 & \cdot \qquad \cdot \;\; \cdot \;\; \cdot & x, y \\ \sigma & \delta_1, \ \delta_1'; \dots; \delta_\sigma, \ \delta_\sigma' & \end{array}\right) = F\left(\begin{array}{c} \beta_1, \dots, \beta_\nu; \ x \\ \delta_1, \dots, \delta_\sigma \end{array}\right) F\left(\begin{array}{c} \beta_1', \dots, \beta_\nu'; \ y \\ \delta_1', \dots, \delta_\sigma' \end{array}\right); \tag{7}$$

$$F\left(\begin{array}{c|c|c} \omega & \alpha_1, \dots, \alpha_\omega & \\ 1 & \beta_1, \ \beta_1' & \\ \omega & \gamma_1, \dots, \gamma_\omega & x, x \\ 0 & \cdot \ \cdot \ \cdot \ \cdot \ \cdot & \end{array}\right) = F\left(\begin{array}{c} \alpha_1, \dots, \alpha_\omega, \ \beta_1 + \beta_1'; \ x \\ \gamma_1, \dots, \gamma_\omega \end{array}\right). \tag{8}$$

Burchnall and Shandy ([4]) gave numerous examples of expansions in double Appell hypergeometric functions. In this article we shall consider certain expansions in Kampé de Fériet functions and shall show how the results of Burchnall and Shandy can be obtained from our general theorems as special cases. As far as I know, expansions in the functions (1) have not been considered in the literature. Burchnall and Shandy introduced a certain differential operator and obtained their results with the aid of this operator. Their arguments were purely symbolic. They also assert on p. 255 of ([4]) that this symbolic method gives no results for the remaining 12 series. With the aid of our method these results are obtained directly and are based on rearrangement of series. The constants and variables are such that the functions under consideration exist.

Theorem 1. The formula is valid

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j; 2r) \prod_{j=1}^{\nu}\{(\beta_j; r)(\beta'_j; r)\}}{r! \prod_{j=1}^{\rho}(\gamma_j; 2r) \prod_{j=1}^{\sigma}\{(\delta_j; r)(\delta'_j; r)\}} \frac{x^r y^r}{(b; r)} \times$$

$$\times F\left(\begin{array}{c|c|c} \mu & \alpha_1+2r, \ldots, \alpha_\mu+2r & \\ \nu & \beta_1+r,\ \beta'_1+r; \ldots; \beta_\nu+r,\ \beta'_\nu+r & \\ \rho & \gamma_1+2r, \ldots, \gamma_\rho+2r & x, y \\ \sigma & \delta_1+r,\ \delta'_1+r; \ldots; \delta_\sigma+r,\ \delta'_\sigma+r & \end{array}\right) \tag{9}$$

$$= F\left(\begin{array}{c|c|c} \mu+1 & \alpha_1, \ldots, \alpha_\mu,\ b & \\ \nu & \beta_1,\ \beta'_1; \ldots; \beta_\nu,\ \beta'_\nu & \\ \rho & \gamma_1, \ldots, \gamma_\rho & x, y \\ \sigma+1 & \delta_1,\ \delta'_1; \ldots; \delta_\sigma,\ \delta'_\sigma;\ b,\ b & \end{array}\right),$$

where $\mu + \nu \leqslant \rho + \sigma + 1$.

Theorem 2. The formula is valid

$$\sum_{r=0}^{\infty} (-1)^r \frac{\prod_{j=1}^{\mu-1}(\alpha_j; 2r) \prod_{j=1}^{\nu}\{(\beta_j; r)(\beta'_j; r)\}}{\prod_{j=1}^{\rho}(\gamma_j; 2r) \prod_{j=1}^{\sigma}\{(\delta_j; r)(\delta'_j; r)\}} \frac{(a; r) x^r y^r}{r!} \times$$

$$\times F\left(\begin{array}{c|c|c} \mu & \alpha_1+2r, \ldots, \alpha_{\mu-1}+2r,\ a+r & \\ \nu & \beta_1+r,\ \beta'_1+r; \ldots; \beta_\nu+r,\ \beta'_\nu+r & \\ \rho & \gamma_1+2r, \ldots, \gamma_\rho+2r & x, y \\ \sigma & \delta_1+r,\ \delta'_1+r; \ldots; \delta_\sigma+r,\ \delta'_\sigma+r & \end{array}\right) \tag{10}$$

$$= F\left(\begin{array}{c|c|c} \mu-1 & \alpha_1, \ldots, \alpha_{\mu-1} & \\ \nu+1 & \beta_1,\ \beta'_1; \ldots; \beta_\nu,\ \beta'_\nu;\ a,\ a & \\ \rho & \gamma_1, \ldots, \gamma_\rho & x, y \\ \sigma & \delta_1,\ \delta'_1; \ldots; \delta_\sigma,\ \delta'_\sigma & \end{array}\right).$$

Let us now consider special cases. Thus, for example, formula (9) in combination with formula (2) gives

$$\sum_{r=0}^{\infty} \frac{(\alpha; 2r)(\beta; r)(\beta'; r)}{r!(b; r)(\gamma; 2r)} x^r y^r F^{[1]}[\alpha+2r;\ \beta+r;\ \beta'+r;\ \gamma+2r;\ x, y] =$$

$$= F\left(\begin{array}{c|c|c} 2 & \alpha,\ b & \\ 1 & \beta,\ \beta' & \\ 1 & \gamma & x, y \\ 1 & b,\ b & \end{array}\right), \tag{11}$$

where $|x| + |y| < 1$. Putting $\beta = \beta' = b$ in (11) and applying (6), we obtain equation (38) of the paper [4]

$$\sum_{r=0}^{\infty} \frac{(\alpha; 2r)(b; r)}{r!(\gamma; 2r)} x^r y^r F^{[1]}[\alpha + 2r;\ b + r,\ b + r;\ \gamma + 2r;\ x, y] =$$

$$= F\left(\begin{matrix}\alpha,\ b;\ x + y\\ \gamma\end{matrix}\right), \tag{12}$$

where $|x| + |y| < 1$.

In the same way, relation (9) in combination with (3) gives

$$\sum_{r=0}^{\infty} \frac{(\alpha; 2r)(\beta; r)(\beta'; r)}{r!(b; r)(\delta; r)(\delta'; r)} x^r y^r F^{[2]}[\alpha + 2r;\ \beta + r,\ \beta' + r;\ \delta + r,\ \delta' + r;\ x, y] =$$

$$= F\left(\begin{array}{c|c|c} 2 & \alpha,\ b & \\ 1 & \beta,\ \beta' & \\ 0 & \cdots & x, y \\ 2 & \delta,\ \delta';\ b,\ b & \end{array}\right), \tag{13}$$

which is also new and is valid in the corresponding region of convergence. Formula (9) and formula (4) give

$$\sum_{r=0}^{\infty} \frac{(\beta_1; r)(\beta_1'; r)(\beta_2; r)(\beta_2'; r)}{r!(b; r)(\gamma; 2r)} x^r y^r F^{[3]}[\beta_1 + r,\ \beta_1' + r;\ \beta_2 + r,\ \beta_2' + r;\ \gamma + 2r;\ x, y] =$$

$$= F\left(\begin{array}{c|c|c} 1 & b & \\ 2 & \beta_1,\ \beta_1';\ \beta_2, \beta_2' & x, y \\ 1 & \gamma & \\ 1 & b,\ b & \end{array}\right). \tag{14}$$

If in formula (14) we put $b = \gamma$ and apply (7), then the following equation is obtained ([4], p. 254, equation (29))

$$\begin{aligned} &\sum_{r=0}^{\infty} \frac{(\beta_1; r)(\beta_1'; r)(\beta_2; r)(\beta_2'; r)}{r!(\gamma; r)(\gamma; 2r)} x^r y^r \times \\ &\times F^{[3]}[\beta_1 + r,\ \beta_1' + r;\ \beta_2 + r,\ \beta_2' + r;\ \gamma + 2r;\ x, y] \\ &= F\left(\begin{matrix}\beta_1, \beta_2;\ x\\ \gamma\end{matrix}\right) F\left(\begin{matrix}\beta_1', \beta_2';\ y\\ \gamma\end{matrix}\right), \end{aligned} \tag{15}$$

where $|x|, |y| < 1$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.