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Abstract

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LOWER BOUNDS BY THE SIEVE METHOD IN ALGEBRAIC NUMBER FIELDS

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§ 1. The transfer of lower bounds to arbitrary algebraic number fields has not been carried out up to now because the scheme of the sieve method in the field of rational numbers does not possess the generality sufficient for it to be transferable to arbitrary fields without substantial changes.

In particular, for the scheme of the paper ([1]) the form of the zero-free region of the Riemann zeta-function $\zeta(s)$ is of fundamental importance:

$$\operatorname{Re} s > 1 - \frac{c_0}{(\ln|t|)^{1-\varepsilon_0}}, \qquad |t| > e.$$

For it it is essential that $\varepsilon_0 > 0$ (it is now known that $\varepsilon_0 = 1/3$). For $\varepsilon_0 = 0$ the scheme ceases to work. For the Dedekind zeta-function $\zeta_K(s)$ of an arbitrary field K the following zero-free region is now known:

$$\operatorname{Re} s > 1 - \frac{c_0}{\ln|t|}, \qquad |t| \geq |d|,$$

where d is the discriminant of K, i.e. precisely the case when the scheme of the paper ([1]) does not work.

Here we shall briefly present a new method of proving Theorem 1 of ([1]), which shows its validity in any field K of degree $n \geq 1$. Since this theorem is central in transferring lower bounds to arbitrary fields, from it, according to the known scheme of B. V. Levin ([2]), there follows the assertion: the binary equation

$$\xi = \alpha + \beta, \tag{1}$$

is solvable, where ξ is any sufficiently large (in norm) integral "even" number of the field K; α and β are integers of the field K with the condition that the principal ideal α has not more than two prime factors and β not more than three.

By an "even" number ξ one should understand the following: if in K there is an ideal $\mathfrak{p}_0$ with $N\mathfrak{p}_0 = 2$, then $\mathfrak{p}_0/\xi$. If there is no such ideal in K, then ξ may be any sufficiently large integer of the field K.

§ 2. For the new proof of Theorem 1 from [1] the following lemma is of essential importance.

Lemma 1. Denote by $N_K(\sigma, T, 2T)$ the number of zeros ρ of the Dedekind zeta-function of the field K of degree n, lying in the rectangle $\operatorname{Re}\rho > \sigma$, $T \leq |\operatorname{Im}\rho| \leq 2T$. Then the estimate

$$N_K(\sigma, T, 2T) \ll T^{(n+4)(1-\sigma)}(\ln T)^{4n+1},$$

is valid, which becomes nontrivial for

$$\sigma > 1 - \frac{1}{n+4}.$$

The proof of this lemma is carried out according to the scheme which is used to obtain density theorems on zeros of the Riemann zeta-function $\zeta(s)$.

Using Lemma 1, we obtain:

Lemma 2. In the region $x > \exp(\sqrt{|d|})$,

$$\operatorname{Re}s > 1 - \frac{0.5c_0}{\sqrt{\ln x}}, \qquad |\operatorname{Im}s| < \exp(\sqrt{\ln x})$$

the equality

$$\prod_{N\mathfrak{p}<x}\left(1 - \frac{1}{(N\mathfrak{p})^s}\right)\zeta_K(s)(s-1) = \frac{e^{-\gamma+\eta(s,x)}}{\ln x}\left(1 + O\left(e^{-a\sqrt{\ln x}}\right)\right),$$

is valid.

where

$$\eta(s,x) = \int_L \frac{x^{1-w}-1}{w-1}\,dw;$$

L is the segment of the straight line joining the points s and 1; moreover, if $\operatorname{Im}s \neq 0$, then the second equality holds:

$$\prod_{N\mathfrak{p}<x}\left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1} = e^{\gamma(s,x)}\zeta_K(s)\left(1 + O\left(e^{-a\sqrt{\ln x}}\right)\right),$$

where

$$\gamma(s,x)=\int_{-\infty}^{1-\sigma}\frac{x^{u-it}}{u-it}\,du,\qquad s=\sigma+it.$$

From this lemma one obtains:

Corollary 1. The Mertens formula in the field K has the form

$$\prod_{N\mathfrak{p}<x}\left(1-\frac{1}{N\mathfrak{p}}\right)=\frac{w\sqrt{|d|}}{2^r(\pi/2)^{r_2}Rh}\,\frac{e^{-c}}{\ln x}\left(1+O\left(e^{-a\sqrt{\ln x}}\right)\right),$$

where the classical constants of the field K enter into the right-hand side of the equality.

Using Lemmas 1 and 2 and Corollary 1, one can prove Theorem 1 of ([1]) for an arbitrary field K.

Theorem 1. Let $f_1(\mathfrak{q})=f(\mathfrak{q})-1$,

$$f(\mathfrak{q})=\begin{cases}\dfrac{N\mathfrak{q}}{2}, & \text{if } \mathfrak{q}\nmid\xi_0,\\ N\mathfrak{q}, & \text{if } \mathfrak{q}\mid\xi_0,\end{cases}$$

$$\Pi_{\mathfrak{p}}=\prod_{N\mathfrak{q}<N\mathfrak{p}}\left(1-\frac{1}{f(\mathfrak{q})}\right),\qquad z_{\mathfrak{p}}=\sqrt{\frac{z}{N\mathfrak{p}}}.$$

Then, if

$$N\mathfrak{p}>\exp\left(\frac{\ln z}{8\ln\ln z}\right),$$

the equality

$$\sum_{N\mathfrak{a}_{\mathfrak{p}}\le z_{\mathfrak{p}}}\frac{\mu^2(\mathfrak{a}_{\mathfrak{p}})}{f_1(\mathfrak{a}_{\mathfrak{p}})}=\frac{1-\varepsilon(u_{\mathfrak{p}})}{\Pi_{\mathfrak{p}}}$$

holds, where

$$\varepsilon(u_{\mathfrak{p}})=-\frac{e^{2\omega}}{2\pi}\int_{-\sqrt{\ln z}}^{\sqrt{\ln z}}\frac{e^{u_{\mathfrak{p}}s}}{s^3}\,e^{2\psi(t)}\,dt+O\left(\frac{e^{-u_{\mathfrak{p}}}}{\sqrt{\ln z}}\right),$$

$$s=-1+it,\qquad u_{\mathfrak{p}}=\left(\frac{\ln z}{\ln N\mathfrak{p}}-1\right),\qquad \omega=\sum_{n=1}^{\infty}\frac{1}{n!\,n},$$

$$\psi(t) = u(t) - iv(t),$$

$$u(t) = e\int_0^t \frac{x\cos x - \sin x}{1+x^2}\,dx, \qquad v(t) = e\int_0^t \frac{x\sin x + \cos x}{1+x^2}\,dx.$$

In proving this theorem, as in ([1]), it becomes necessary to investigate the integral of the product $\Pi_1(s, \mathfrak{p})$ over a contour that lies in the half-plane $\operatorname{Re} s \le 0$. We define this contour as follows. It passes through two points s_0 and $\bar{s}_0$, where

$$s_0 = -\frac{0.5c_0}{\sqrt{\ln N\mathfrak{p}}} + it_0,$$

t_0 is determined from the condition

$$t_0 \ln N\mathfrak{p} = e^{0.5c_0\sqrt{\ln N\mathfrak{p}}}.$$

The study of the integral along that part of the contour which is defined by the condition $|\operatorname{Im} s| \le t_0$ is carried out exactly as in ([1]) (since Lemma 1 already begins to apply on this segment), with the replacement of the corresponding properties of $\zeta(s)$ by the more general properties of $\zeta_K(s)$.

§ 3. The investigation of the integral over the part of the contour with the condition $|\operatorname{Im} s| > t_0$ requires an entirely different technique. This part of the contour must be drawn along the line

$$\left(-\frac{0.5c_0}{\sqrt{\ln N\mathfrak{p}}} + it\right)$$

with a detour around each zero of $\zeta_K(1+s)$ which has real part satisfying

$$\operatorname{Re}\rho > 1 - \frac{c_0}{\sqrt{\ln N\mathfrak{p}}},$$

over a segment of length $2\sqrt{|\rho|}$ lying on the imaginary axis.

The measure of such segments lying on the imaginary axis in the interval $(T, 2T)$ will not exceed

$$2\sqrt{T} N_K\left(1 - \frac{c_0}{\sqrt{\ln N\mathfrak{p}}},\, T,\, 2T\right).$$

Applying Lemma 1, we obtain an estimate for the part of our integral over these segments. This part does not exceed a quantity of order $\exp(-a\sqrt{\ln N\mathfrak{p}})$. On the remaining part of the contour (after removal of the segments lying on the imaginary axis), Lemma 2 is again valid. This can be shown with the help of Lemma 1. Using the known properties of $\zeta_K(1+s)$, one can already obtain an estimate for the part of our integral over the contour with the condition $|\operatorname{Im} s| > t_0$. It has magnitude of order

$$\exp(-a\sqrt{\ln N\mathfrak{p}}).$$

Hence the validity of Theorem 1 follows. We note that, for the investigation of the part of the integral on the segment $-\sqrt{\delta} \le t \le \sqrt{\delta}$, $\delta = \frac{1}{\ln N\mathfrak{p}}$, which gives the main part in the formula for $\varepsilon(u_\mathfrak{p})$, it is better to use the first equality of Lemma 2, since the function $\eta(1+s, x)$ is considerably easier to reduce to the function $\psi(t)$ with the extraction of Euler' s constant and ω than the function $\gamma(1+s, x)$.

By this method one can obtain a lower estimate for the number of solutions of equation (1):

$$A_1(\xi, K) \ge c_1' \frac{w^2 |d|^{3/2} \mathfrak{S}_K(\xi)}{2^n (2\pi)^{r_2} (Rh)^2} \frac{|N\xi|}{\ln^2 |N\xi|},$$

where $c_0' > 0$ is an absolute constant,

$$\mathfrak{S}_K(\xi) = \prod_{\mathfrak{p}|\xi} \left(1 - \frac{1}{N\mathfrak{p}}\right)^{-1} \prod_{\mathfrak{p}\nmid\xi} \left(1 - \frac{2}{N\mathfrak{p}}\right) \left(1 - \frac{1}{N\mathfrak{p}}\right)^{-2}.$$

For the case of the difference problem

$$\xi_0 = \alpha - \beta, \qquad |\alpha|,\ |\beta| \le |\xi|,$$

we obtain a completely analogous estimate with the replacement of the singular series by $\mathfrak{S}_K(\xi_0)$.

Moreover, for the case of the binary problem in an arbitrary field K

$$\xi = \alpha + \beta, \qquad (\alpha) = \mathfrak{p}_1, \qquad (\beta) = \mathfrak{p}_2, \tag{2}$$

we obtain an upper estimate for the number of solutions of equation (2):

$$A(\xi, K) \le 8 \frac{w^2 |d|^{3/2} \mathfrak{S}_K(\xi)}{2^n (2\pi)^{r_2} (Rh)^2} \frac{|N\xi|}{\ln^2 |N\xi|} + O\left(\frac{|N\xi|}{(\ln |N\xi|)^{2.5}}\right).$$

An analogous estimate is valid for generalized "twins" in K:

$$\xi_0 = \alpha - \beta, \qquad (\alpha) = \mathfrak{p}_1, \qquad (\beta) = \mathfrak{p}_2, \qquad |\alpha|,\ |\beta| \le |\xi|,$$

with the replacement of $\mathfrak{S}_K(\xi)$ by $\mathfrak{S}_K(\xi_0)$.

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REFERENCES

1. A. I. Vinogradov, *Mat. sborn.*, **41** (83), No. 1, 3 (1957).

2. B. V. Levin, *Dokl. AN UzSSR*, No. 11 (1962).

Note: Figure translations are in progress. See original paper for figures.