



Soviet-era science, translated into English

Yu. K. SUETIN

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.72794>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Yu. K. SUETIN

ON THE CONSTANTS OF CONVERGENCE AND UNIQUENESS FOR CERTAIN INTERPOLATION PROBLEMS

(Presented by Academician M. V. Keldysh on 8 VI 1964)

The paper considers interpolation problems, special cases of which are the Abel-Goncharov interpolation problem, the Pommiez problem ⁽¹⁾, and the (γ) -problems introduced by M. M. Dzhragilev and O. P. Chukhova ⁽²⁾.

Let $m(x)$ be an arbitrary function defined on the set $X = [1, +\infty]$ and satisfying the conditions $m(x) \uparrow +\infty$, $m(1) = 1$, $m'(x)/m(x) \downarrow 0$. Denote by $A_r^\alpha(m)$ the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{v_n^\alpha} z^n, \quad v_n = \prod_{p=1}^n m(p), \quad \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \leq r,$$

$$0 < r < \infty, \quad 0 \leq \alpha < \infty.$$

Let γ , $0 \leq \gamma < \infty$, be fixed and let $M_\alpha^\gamma \supset \{z_k\}$ be an arbitrary sequence of complex numbers satisfying the inequalities

$$|z_k| \leq [m(k+1)]^{\alpha-\gamma}, \quad k = 0, 1, \dots$$

By a (γ) -problem with interpolation nodes $\{z_k\} \subset M_\alpha^\gamma$ we shall mean any interpolation problem generated by a system of functionals L_n , $n = 0, 1, 2, \dots$, defined on $\{z^k\}$ by the equalities

$$L_n(z^k) = \begin{cases} 0, & n > k, \\ z_n^{k-n} \left(\frac{v_k}{v_n v_{k-n}} \right)^\gamma, & n \leq k. \end{cases} \quad (1)$$

If $m(p) = p$, then for $\gamma = 1$ we have the Abel-Goncharov problem; for $\gamma = 0$, the Pommiez problem; and for γ , $0 \leq \gamma < \infty$, the (γ) -problems of ⁽²⁾.

The **boundary of uniqueness** of the set M_α^γ in the (γ) -problem will be called the lower bound $W_\alpha^\gamma(m)$ of those values r for which there exist a function $f(z) \in A_r^\alpha(m)$ and a sequence $\{z_k\} \subset M_\alpha^\gamma$ such that

$$L_n(f) = 0, \quad n = 0, 1, 2, \dots, \quad f(z) \not\equiv 0.$$

Define polynomials $\{P_n(z)\}$ by the conditions

$$L_k(P_n) = \begin{cases} 0, & n \neq k, \\ 1, & n = k; \end{cases} \quad n, k = 0, 1, 2, \dots$$

The **boundary of convergence** of the set M_α^γ in the (γ) -problem will be called the upper bound $S_\alpha^\gamma(m)$ of those values r for which every function $f(z) \in A_r^\alpha(m)$ is representable by the series

$$f(z) = \sum_{n=0}^{\infty} L_n(f)P_n(z) \quad (2)$$

for any $\{z_k\} \subset M_\alpha^\gamma$.

In every (γ) -problem two basic problems arise:

1. To find the class of analytic functions (in the sense of growth for entire functions and in the sense of the radius of the disk of convergence for analytic functions) for which the series (2) converges to $f(z)$, i.e. to find $S_\alpha^\gamma(m)$ (the convergence problem).
2. To find the class of analytic functions for which the condition $L_n(f) = 0$, $n = 0, 1, \dots$, implies $f(z) \equiv 0$, i.e. to find $W_\alpha^\gamma(m)$ (the uniqueness problem).

None of the quantities $W_\alpha^\gamma(m)$, $S_\alpha^\gamma(m)$ has been found, and only for the classical case $\gamma = 1$, $m(p) = p$ (the Abel–Goncharov problem) and the case $\alpha = \gamma = 0$ (the Pommier problem) are estimates of these quantities available (3–7).

On the basis of these estimates, R. Boas (8) put forward the conjecture that $W_0^1(p)$ and $W_1^1(p)$ coincide (the Whittaker and Goncharov constants). Later M. A. Evgrafov formulated the more general hypothesis that all constants in the Abel–Goncharov problem coincide and proved (9) that $W_1^1(p) = S_1^1(p)$.

M. M. Dragilev (10) proved that $S_0^1(p) = S_1^1(p)$, while in (2) the equalities $S_\alpha^\gamma(p) = S_\gamma^\gamma(p)$ were proved. Thus, in the case $m(p) = p$ the convergence constants $S_\alpha^\gamma(p)$ of the problems do not depend on α . For the uniqueness constants the analogous problem (in particular, Boas' s problem) remained unsolved. In the present paper the equalities

$$S_\alpha^\gamma(m) = S_\gamma^\gamma(m) = W_\gamma^\gamma(m) = W_\alpha^\gamma(m) \quad (3)$$

are proved for arbitrary α , $0 \leq \alpha < \infty$, and fixed γ , $0 \leq \gamma < \infty$; consequently, this problem is also solved for the uniqueness constants. This proves the validity of the conjectures of M. A. Evgrafov and R. Boas.

Let

$$P_n(0) = (-1)^n v_n^\gamma \alpha_n(z_0, \dots, z_{n-1}),$$

$$d_n(\alpha, \gamma, m) = v_n^{\gamma-\alpha} \max_{z_i \in M_\alpha^\gamma} |\alpha_n(z_0, \dots, z_{n-1})|,$$

where γ , $0 \leq \gamma < \infty$, is fixed, and α , $0 \leq \alpha < \infty$, is arbitrary. Then the following lemmas hold.

Lemma 1. There exists the upper limit

$$\overline{\lim}_{n \rightarrow \infty} d_n^{1/n}(\alpha, \gamma, m) = d(\alpha, \gamma, m), \quad 1 \leq d(\alpha, \gamma, m) \leq 2.$$

For the proof one uses the recurrence relation for $P_k(z)$, $k = 0, 1, \dots, n$,

$$\sum_{k=0}^n z_k^{n-k} \left(\frac{v_n}{v_k v_{n-k}} \right)^\gamma P_k(z) = z^n,$$

which is nothing other than the expansion of z^n in terms of $\{P_k(z)\}$.

Lemma 2.

$$2d_n(\alpha, \gamma, m) \geq d_k(\alpha, \gamma, m) \max_{z_i \in M_\alpha^\gamma} |\alpha_{n-k}(z_k, \dots, z_{n-1})| \left(\frac{v_n}{v_k} \right)^{\gamma-\alpha},$$

$$k = 0, 1, \dots, n; \quad n = 0, 1, \dots$$

The basis of the proof is the equality

$$\begin{aligned} \alpha_n(z_0, \dots, z_{n-1}) - \alpha_n(z_0, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_{n-1}) \\ = \alpha_k(z_0, \dots, z_{k-1}) \alpha_{n-k}(z_k, \dots, z_{n-1}), \end{aligned} \quad (4)$$

which is easily obtained from the representation of $P_n(z)$ in determinant form, analogous to that noted in ⁽¹¹⁾ (p. 127) for the Abel–Goncharov polynomial.

Lemma 3.

$$d_{n-1}(\alpha, \gamma, m) \leq d_n(\alpha, \gamma, m), \quad n = 1, 2, 3, \dots \quad (5)$$

The proof is trivial if one observes that, according to (4),

$$d\alpha_n(z_0, \dots, z_{n-1})/dz_{n-1} = \alpha_{n-1}(z_0, \dots, z_{n-2}).$$

Lemma 4. If

$$d_n^{1/n}(\alpha, \gamma, m) = d_{n-1}^{1/(n-1)}(\alpha, \gamma, m) - \Delta_n, \quad \Delta_n > 0,$$

then

$$\Delta_n < A/n, \quad A = \text{const}, \quad n > N.$$

The assertion follows from (5) and Lemma 1.

Lemma 5. For any $\varepsilon > 0$ there exist arbitrarily large m_0 and n_0 such that, for $p = 1, 2, 3, \dots$,

$$\begin{aligned} \max_{z_i \in M_\alpha^\gamma} |\alpha_{m_0}(z_{n_0+(p-1)m_0}, \dots, z_{n_0+pm_0-1})| \left(\frac{\nu_{n_0+pm_0}}{\nu_{n_0+(p-1)m_0}} \right)^{\gamma-\alpha} > \\ > [d(\gamma, \gamma, m) - \varepsilon/3]^{m_0}, \end{aligned} \quad (6)$$

The proof in its general outline uses one device from (2).

Lemma 6.

$$d_n^{1/n}(\alpha, \gamma, m) > d(\gamma, \gamma, m) - \varepsilon, \quad n > N(\varepsilon).$$

The proof is based on Lemma 4 and the inequalities

$$d_{n_0+pm_0}^{1/(n_0+pm_0)}(\alpha, \gamma, m) > d(\gamma, \gamma, m) - \frac{\varepsilon}{2} \quad \text{for } p > N(\varepsilon). \quad (7)$$

To prove (7) it is enough to put in Lemma 2 $n = n_0 + m_0, n_0 + 2m_0, \dots, n_0 + pm_0$; $k = n_0, n_0 + m_0, \dots, n_0 + (p-1)m_0$, multiply the resulting inequalities and estimate from below the quantities on the right, taking (6) into account.

From Lemmas 1 and 6 there follow the inequalities

$$\frac{1}{2d_n(\alpha, \gamma, m)} \max_{z_i \in M_\alpha^\gamma} |\alpha_{n-k}(z_k, \dots, z_{n-1})| \left(\frac{\nu_n}{\nu_k} \right)^{\gamma-\alpha} \leq \frac{1}{[d(\gamma, \gamma, m) - \varepsilon]^k}, \quad k > N(\varepsilon). \quad (8)$$

Theorem 1. For every $r > 1/d(\gamma, \gamma, m)$ there exist a function $f(z) \in A_r^\alpha$, $0 \leq \alpha < \infty$, and a sequence $\{z_k\} \subset M_\alpha^\gamma$, $k = 0, 1, \dots$, such that

$$L_n(f) = 0, \quad n = 0, 1, 2, \dots, \quad f(z) \not\equiv 0.$$

Theorem 2. For $r = 1/d(\gamma, \gamma, m)$ there exist a function $f(z)$, analytic in the disk $|z| < 1/r$, and a sequence $\{z_k\} \subset M_0^\gamma$, $k = 0, 1, \dots$, such that

$$L_n(f) = 0, \quad n = 0, 1, 2, \dots, \quad f(z) \not\equiv 0.$$

Theorem 3. For any $r < 1/d(\gamma, \gamma, m)$ every function $f(z) \in A_r^\alpha(m)$, $0 \leq \alpha < \infty$, can be expanded in a series in the interpolation polynomials $\{P_n(z)\}$ of problem (γ), whatever the interpolation nodes $\{z_k\} \subset M_\alpha^\gamma$ may be.

In constructing $f(z)$ for the first two theorems, the inequalities (8) and the idea of M. A. Evgrafov (⁹, pp. 111–113) are used.

The proof of Theorem 3 in its general outline reproduces the proof of part a) of Theorem 3 (²) and rests on the matrix criterion for a basis (¹², ¹⁷) in a space of analytic functions and on a lemma of M. M. Dragilev (¹⁰). The validity of (3) follows from Theorems 1, 2 and 3, the definitions of $W_\alpha^\gamma(m)$, $S_\alpha^\gamma(m)$, and the obvious inequality $S_\alpha^\gamma(m) \leq W_\alpha^\gamma(m)$.

Let α, γ , $0 \leq \alpha \leq \infty$, $0 \leq \gamma < \infty$, be arbitrary, and let a function $\varphi(m)$ be defined on the set of natural numbers and satisfy the conditions

$$\lim_{m \rightarrow \infty} \varphi(m) = \alpha, \quad \lim_{m \rightarrow \infty} \frac{m^{\varphi(m)}}{(m+1)^{\varphi(m+1)}} = 1, \quad \lim_{m \rightarrow \infty} m^{\gamma - \varphi(m)} = 1, \quad \text{if } \alpha = \gamma,$$

$$\lim_{m \rightarrow \infty} m^{\varphi(m)} = +\infty, \quad \text{if } \alpha = 0, \quad \varphi(m) \neq 0,$$

$$m^{|\varphi(m) - \gamma|} \leq (m+1)^{|\varphi(m+1) - \gamma|}, \quad \text{if } \alpha \neq \gamma, \quad m > N.$$

Denote by A_r^φ the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{\nu_n} z^n, \quad \nu_n = \prod_{m=1}^n m^{\varphi(m)}, \quad \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \leq r, \quad 0 < r < \infty,$$

and by $T_\varphi^\gamma \ni \{z_k\}$, $k = 0, 1, 2, \dots$, an arbitrary sequence of complex numbers satisfying the inequalities

$$|z_k| \leq (k+1)^{\varphi(k+1) - \gamma}.$$

We define the functionals by the conditions (1), putting $m(p) = p$, and introduce for the (γ) -problems the constants of convergence S_φ^γ and uniqueness W_φ^γ . Then the equalities $W_\varphi^\gamma = W_\gamma^\gamma = S_\gamma^\gamma = S_\varphi^\gamma$ hold, independently of the choice of the functions $\varphi(m)$.

Let:

1) A_r^α be the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)^\alpha} z^n, \quad \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \leq r, \quad 0 < r < \infty, \quad 0 \leq \alpha < \infty.$$

2) $M_\alpha^\gamma \supset \{z_k\}$ be a sequence of points satisfying the inequalities

$$|z_k| \leq (k+1)^{\alpha-\gamma}, \quad k = 0, 1, \dots$$

(γ , $0 \leq \gamma < \infty$, is fixed).

3) L_n , $n = 0, 1, \dots$, be functionals.

4) $T = \{m_n\}$ be some sequence of natural numbers.

Denote by $W_\alpha^\gamma(T)$ the lower bound of those values r for which there exist a function $f(z) \in A_r^\alpha$ and a system of finite sequences $\{z_n^{(r)}\} \subset M_\alpha^\gamma$ such that

$$L_n(f) = 0, \quad |z_n^{(i)}| \leq (n+1)^{\alpha-\gamma}, \quad i = 1, \dots, m_n, \quad n = 0, 1, \dots, \quad f(z) \not\equiv 0.$$

It is necessary to estimate $W_\alpha^\gamma(T)$. For $\gamma = 1$ the problem under consideration is Pólya's problem¹³; for $\gamma = \alpha = m_n \equiv 1$ it is S. N. Bernstein's problem¹⁴, and Erdős and Rényi^{15,16} proved for $m_n = p = \text{const}$ that $W_1^1(p) \geq p/e$; the author¹⁸ proved the validity of the inequality

$$2p \ln \rho \leq \ln I_0 [2W_1^1(p)\rho] \quad \text{for any } \rho > 1 \quad (9)$$

($I_0(x)$ is the Bessel function of zero order of imaginary argument). Letting $p \rightarrow \infty$ in (9), putting $\rho = e$, and using the trivial asymptotic expression for $I_0(x)$, we obtain $W_1^1(p) > p/e + \ln p/4e + O(1)$. This result is repeated in Wilf's paper¹⁹.

It is not difficult to obtain, by the method of¹⁸, the more general result

$$2p \ln \rho \leq \ln \Phi_\gamma [W_\gamma^\gamma(p)\rho]$$

for any $\rho > 1$, $0 < \gamma < \infty$, where

$$\Phi_\gamma(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(k!)^{2\gamma}}.$$

Hence, using the asymptotic behavior of $\Phi_\gamma(x)$ ²⁰, p. 172, and putting $\rho = e$, we have

$$[W_\gamma^\gamma(p)e]^{1/\gamma} > \frac{p}{\gamma} + \frac{2\gamma-1}{4\gamma} \ln p + O(1) \quad \text{for } \frac{1}{2} \leq \gamma < \infty,$$

$$[W_\gamma^\gamma(p)e]^{1/\gamma} \geq \frac{p}{\gamma} \quad \text{for } 0 < \gamma < \frac{1}{2}.$$

I take this opportunity to express my gratitude to M. A. Evgrafov for posing the problems and for his constant attention to the work.

Moscow State University
named after M. V. Lomonosov

Received
28 V 1964

REFERENCES

- ¹ M. R. Pommiez, C. R., **250**, No. 7, 1168 (1960).
- ² M. M. Dzhragilev, O. P. Chukhilova, Siberian Math. J., **4**, No. 2, 287 (1963).
- ³ N. Levinson, Duke Math. J., **11**, 721 (1944).
- ⁴ N. Levinson, Duke Math. J., **12**, 335 (1945).
- ⁵ S. Macintyre, J. Lond. Math. Soc., **22**, 305 (1947).
- ⁶ S. Kakeya, Proc. Phys.-Math. Soc. Japan, **14** (1932).
- ⁷ M. M. Dzhrbashyan, Izv. Akad. Nauk SSSR, Ser. Mat., **16**, 225 (1952).
- ⁸ R. P. Boas, Duke Math. J., **15**, No. 4, 953 (1948).
- ⁹ M. A. Evgrafov, *The Abel-Goncharov interpolation problem*, Moscow-Leningrad, 1954.
- ¹⁰ M. M. Dzhragilev, Uspekhi Mat. Nauk, **15**, issue 3 (93), 171 (1960).
- ¹¹ M. A. Evgrafov, Trudy Moskov. Mat. Obshch., **5**, 90 (1956).
- ¹² J. M. Whittaker, *Sur les séries de polinomes quelconques*, Paris, 1949.
- ¹³ C. Polya, Bull. Am. Math. Soc., **49**, 178 (1943).
- ¹⁴ S. N. Bernstein, Proceedings of the International Mathematical Congress, Bologna, 1930, **2**, p. 267.
- ¹⁵ P. Erdős, A. Rényi, Acta Math. Acad. Sci. Hung., **7**, No. 2, 125 (1956).
- ¹⁶ P. Erdős, A. Rényi, Acta Math. Acad. Sci. Hung., **8**, No. 1-2, 223 (1957).
- ¹⁷ M. G. Khaplanov, Dokl. Akad. Nauk SSSR, **80**, No. 1, 21 (1951).
- ¹⁸ Yu. K. Suetin, Uspekhi Mat. Nauk, **17**, 3 (105), 197 (1962).
- ¹⁹ H. S. Wilf, Proc. Am. Math. Soc., **14**, No. 2, 238 (1963).
- ²⁰ M. A. Evgrafov, *Asymptotic estimates and entire functions*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.