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1964

SovietRxiv

Abstract

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CONTINUA POSSESSING THE FIXED-POINT PROPERTY

(Presented by Academician P. S. Aleksandrov, 23 X 1963)

1. The problem, posed by P. S. Aleksandrov, of the existence of a fixed point under an arbitrary continuous mapping into itself of an acyclic one-dimensional continuum remains unsolved. The strongest result in this direction is the theorem on the existence of a fixed point under an arbitrary continuous self-mapping of every snake-like continuum (O. V. Lokutsievskii (^{1}), Hamilton (^{2})) and the theorem on the existence of a fixed point under an arbitrary continuous self-mapping of a linearly connected, one-dimensional, acyclic continuum, proved by K. Borsuk in 1954 (^{3}).

Here theorems on the existence of a fixed point are proved for certain one-dimensional acyclic continua.

2. **Theorem 1.** *If a continuum K can be ε-mapped onto a finite tree T_ε, $g_\varepsilon : K \to T_\varepsilon$, in such a way that the inverse images of the branching points of the tree are connected, then the continuum K possesses the fixed-point property.*

The proof is based on the following lemmas.

Lemma 1. *Let $g_\varepsilon : K \to T_\varepsilon$ be an arbitrary ε-mapping of the continuum K onto a finite tree T_ε, and let $\widehat{pq}$ be an arc of the tree T_ε such that inside $\widehat{pq}$, i.e. in $\widehat{pq} \setminus (p \cup q)$, there are no branching points of the tree T_ε. Then in the compactum*

$$C = g_\varepsilon^{-1}(\widehat{pq})$$

there exists a continuum A such that $g_\varepsilon(A) = \widehat{pq}$.

The proof of Lemma 1 is carried out by elementary set-theoretic methods.

Let $\operatorname{komp}_x X$ denote the component of the point x in the set X.

Definition 1. We say that a point x belonging to the continuum A of the preceding lemma **goes to the left** under the mapping f, if one of the following two conditions is satisfied:

1) $g_\varepsilon(x) \neq q$ and

$$g_\varepsilon f(x) \in T_\varepsilon \setminus \operatorname{komp}_q(T_\varepsilon \setminus g_\varepsilon(x));$$

2) $g_\varepsilon(x) = q$ and

$$g_\varepsilon f(x) \in \overline{\operatorname{komp}_p(T_\varepsilon \setminus g_\varepsilon(x))}.$$

The point x **goes to the right** if in 1) and 2) one replaces q by p.

Lemma 2. *If in the continuum A of Lemma 1 there exist two points x and y such that under the continuous mapping f the point x goes to the left and the point y goes to the right, then in the continuum A there exists a point $a \in A$ such that*

$$g_\varepsilon(a) = g_\varepsilon f(a).$$

Proof. Every point $a \in A$ goes either to the left or to the right, and the set of points going to the left (to the right) is closed. The continuum A is the sum of two nonempty closed sets, and for every point a in their intersection

$$g_\varepsilon(a) = g_\varepsilon f(a).$$

To prove Theorem 1 it is enough to show that there is a continuum A satisfying the conditions of Lemmas 1 and 2. This can be done by using the connectedness property of the inverse images of the branching points of the tree T_ε. Therefore

there exists a point $a \in K$ such that $g_\varepsilon(a) = g_\varepsilon f(a)$, and since g_ε is an ε-mapping, $\rho(a, f(a)) < \varepsilon$. In view of the arbitrariness of ε, the assertion of Theorem 1 follows.

From Theorem 1 the previously known theorems easily follow:

Corollary 1. *Every snakelike continuum has the fixed-point property.*

Corollary 2. *A tree (a locally connected one-dimensional continuum containing no circle) has the fixed-point property, since every tree can be monotonically ε-mapped onto a finite tree.*

3. Let the continuum K be the sum of two continua: $K = K_1 \cup K_2$, where $K_1 \cap K_2$ is a continuum and each of the continua K_1, K_2 has the fixed-point property. It is unknown whether, in this case, the continuum K has the fixed-point property.

If $K_1 \cap K_2$ is a point, then the problem is solved simply. Let $f : K \to K$ be an arbitrary continuous mapping of the continuum K into itself. Let $K_1 \cap K_2 = x$; if $f(x) = x$, then a fixed point exists; suppose $f(x) \in K_2 \setminus x$, and let $\varphi : K \to K$ be a mapping identical on K_2 and sending each point $y \in K_1 \setminus x$ to the point x. Then, for the mapping $\varphi f : K_2 \to K_2$, by hypothesis there exists a point $z \in K_2$ such that $\varphi f(z) = z$. But $z \neq x$, since $\varphi f(x) = f(x) \neq x$; hence $z \in K_2 \setminus x$ and $\varphi f(z) = f(z) = z$.

Theorem 2. *If the continuum K is the sum of two of its proper subcontinua: $K = K_1 \cup K_2$ and $K_1 \cap K_2 = T$ is a tree, and if both continua K_1 and K_2 are one-dimensional and each of them has the fixed-point property, then the continuum K has the fixed-point property.*

Proof. If the continuum K contained a circle, then the circle would lie in one of the continua K_i, $i = 1, 2$; but then the continuum K_i would not have the fixed-point property, and therefore K contains no circle.

Consider two cases:

1) $f(T) \cap T = \Lambda$; $f(T) \subset K_2 \setminus T$. Every tree T is an absolute retract; hence there exists a continuous mapping $\varphi : K_1 \to T$ (a retraction), identical on T. Extend the mapping φ to the whole continuum K, making it identical on K_2. Then the mapping $\varphi f : K_2 \to K_2$ is a mapping of the continuum K_2 into itself. By hypothesis there exists a fixed point for this mapping: $x \in K_2$, $\varphi f(x) = x$. The point $x \in T$, since for $x' \in T$, $f(x') \in K_2 \setminus T$ and $\varphi f(x') = f(x') \neq x'$. Let $f(x) = y$. If $y \in K_1$, then $\varphi(y) \in T$ and, consequently, $\varphi(y) \neq x$. If, however, $y \in K_2 \setminus K_1$, then $\varphi(y) = y$ and, consequently, $\varphi f(x) = f(x) = x$; but then the point x will be fixed also under the mapping $f : K \to K$.

2) $f(T) \cap T \neq \Lambda$. We shall give here only the idea of the proof. $f(T)$ is a one-dimensional locally connected continuum containing no circle; hence $f(T)$ is a tree. $f(T) \cap T = B$ is connected, since K contains no circle. Define a mapping $\psi : f(T) \to B$ (a contraction) as follows: $\psi f(x) = f(x)$ if $f(x) \in B$, and each component $f(T) \setminus B$ is mapped to the point

$$x = B \cap \overline{\text{comp}(f(T) \setminus B)}.$$

The mapping $\psi f : T \to B$ is a mapping of the tree T into itself.

It is first proved that if, under the mapping $\psi f : T \to B$, there is more than one fixed point, then the mapping $f : K \to K$ also has a fixed point. Suppose the mapping $\psi f : T \to B$ has only one fixed point m: $\psi f(m) = m$. If $f(m) \in T$, then $f(m) \in B$ and $\psi f(m) = f(m) = m$.

Now suppose that $f(m) \in K_2 \setminus T$. Define the following mapping:

1) $\psi' : (f(T) \cap K_1) \to B$ is a contraction and is defined analogously to $\psi : f(T) \to B$.

2) ψ' is a homeomorphism on the set $K_1 \setminus (f(T) \cap K_1)$.

3) ψ' is the identity mapping on K_2; then $\psi' : K \to K_2 \cup C$, where $C = \psi'(K_1)$.

Define the mapping (retraction) $\varphi : (K_2 \cap C) \to K_2$, identical on K_2 and such that $\varphi(C) = T$.

Consider the continuous mapping $\varphi\psi' f : K_2 \to K_2$; by the hypothesis, it has a fixed point $\varphi\psi' f(x) = x$. This point is also fixed under the mapping f: $f(x) = x$.

Received
2 VII 1963

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Note: Figure translations are in progress. See original paper for figures.

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