ON A REPRESENTATION OF GENERALIZED ANALYTIC AUTOMORPHIC FUNCTIONS

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ON A REPRESENTATION OF GENERALIZED ANALYTIC AUTOMORPHIC FUNCTIONS

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Let, on the fundamental polygon R of some Fuchsian group S of fractional-linear transformations with boundary L, the equation

$$\frac{\partial U}{\partial \bar{z}} = A(z)\overline{U}, \tag{1}$$

be given, in which $A(z)$ is covariant with respect to $\bar{z}$, and is discontinuous everywhere on R, except for a finite number of points and lines of discontinuity.

In the present note we obtain, for any automorphic solution $U(z)$ of equation (1), having at the prescribed points $z_1, z_2, \dots, z_m$ poles of orders not exceeding $\lambda_1, \lambda_2, \dots, \lambda_m$, respectively, an analogue of the representation of the first kind for generalized analytic functions (^{1}). We shall regard the points $z_1, z_2, \dots, z_m$ as poles of some automorphic function belonging to the group S.

In order that a solution of equation (1) be automorphic with respect to the group S under consideration, it is necessary and sufficient that the condition

$$U^+[\alpha(t)] = U^+(t), \qquad t \in L, \tag{2}$$

be satisfied, where the function $\alpha(t)$ carries out the involutory transformation of the contour L onto itself. The function $\alpha(t)$ is completely determined by the set of generating transformations of the group and on each side of the polygon coincides with the fractional-linear transformation carrying this side into the side equivalent to it, and therefore has, in the general case, discontinuities of the first kind at the vertices t_j of the fundamental polygon.

It is known (^{1}) that every meromorphic solution of equation (1) admits the representation

$$U(z) = \exp[\omega(z)]\varphi(z), \tag{3}$$

where

$$\omega(z) = -\frac{1}{\pi} \iint\limits_R \frac{A_\zeta}{\zeta - z} \frac{\overline{U(\zeta)}}{U(\zeta)}\, dT,$$

and $\varphi(z)$ is an arbitrary meromorphic function. Therefore, substituting (3) into (2), we obtain

$$\varphi^+[\alpha(t)] = \exp\{\omega(t) - \omega[\alpha(t)]\}\varphi^+(t), \qquad t \in L. \tag{4}$$

We solve the boundary-value problem (4), regarding the coefficient $G(t) = \exp\{\omega(t) - \omega[\alpha(t)]\}$ as a known quantity. The function $G(t)$, obviously, has the property

$$G(t)G[\alpha(t)] = 1 \tag{5}$$

and everywhere on L, except perhaps for the vertices t_j, where in the general case it has discontinuities of the first kind, it satisfies the Hölder condition.

It is easy to verify that the function

$$\exp[\Gamma(z)] =$$

$$= \exp\left\{\frac{1}{2\pi i}\int_L \left[\frac{1}{\tau - z} + \sum_k \frac{1}{\tau - S_k(z)} - \frac{1}{\tau - z_0} - \sum_k \frac{1}{\tau - S_k(z_0)}\right] \varphi(\tau)\, d\tau\right\},$$

where $\varphi(t)$ is a solution of the Fredholm integral equation

$$\varphi(t) + \frac{1}{4\pi i}\int_L \left\{\frac{1}{\tau - t} + \sum_k \frac{1}{\tau - S_k(t)} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)} - \sum_k \frac{\alpha'(\tau)}{\alpha(\tau) - S_k[\alpha(t)]}\right\} \varphi(\tau)\, d\tau =$$

$$= \frac{1}{2}[\omega(\alpha(t)) - \omega(t)],$$

and $S_k(z)$ are those transformations of the group S for which the polygons $S_k(R)$ have with R either a common side or a common vertex, satisfies the boundary condition (4). The point $z = z_0$ is arbitrary in R. The function $\exp\{\Gamma(z)\}$, obviously, is holomorphic in R by virtue of the equality

$$-\frac{1}{4\pi i} \ln \frac{G(t_j - 0)}{G(t_j + 0)} = 0,$$

which follows from (5), is bounded at the vertices t_j, and is piecewise holomorphic on the whole plane E with the line of jumps L. Thus, the problem of determining the function $\varphi(z)$ reduces to the construction in R of such an analytic function

$$\psi(z) = \frac{\varphi(z)}{\exp[\Gamma(z)]},$$

which satisfies the boundary condition

$$\psi^+[\alpha(t)] = \psi^+(t) \quad \text{on } L$$

and has at the points $z_1, z_2, \dots, z_m$ poles of orders not exceeding $\lambda_1, \lambda_2, \dots, \lambda_m$, respectively. But such a function is a simple automorphic function belonging to the group S, and can be constructed by the Weierstrass method (^{2}); it admits the representation

$$\psi(z) = c + \sum_{j=1}^{m} \left[c_{j,0} H_0(z, z_j) + c_{j,1} H_1(z, z_j) + \dots + c_{j,\lambda_j - 1} H_{\lambda_j - 1}(z, z_j)\right], \qquad (6)$$

in which the constants $c_{j,0}, c_{j,1}, \dots, c_{j,\lambda_j - 1}$ are connected by the homogeneous system of algebraic equations

$$\sum_{j=1}^{m} \left(c_{j,0} h^j_{\sigma,0} + c_{j,1} h^j_{\sigma,1} + \dots + c_{j,\lambda_j - 1} h^j_{\sigma,\lambda_j - 1}\right) = 0.$$

Here the functions $H_\mu(z, z_j)$ $(\mu = 0, 1, \dots, \lambda_j - 1)$ are the coefficients of the expansions

$$H(z, \tau) f_1'(\tau) = -\sum_{\mu} H_\mu(z, z_j)(\tau - z_j)^\mu$$

of the analogue $H(z, \tau) f_1'(\tau)$ of the Cauchy kernel for the fundamental polygon R (^{3}), and the numbers $h^j_{\sigma,\mu}$ are constants from the expansions

$$H_\mu(z, z_j) = -h^j_{\sigma,\mu}(z - a_\sigma)^{-1} + P(z - a_\sigma),$$

in which the points a_σ $(\sigma = 1, 2, \dots, \rho)$, where ρ is the genus of the fundamental polygon R, are fixed poles of the function $H(z, \tau)$.

Now, combining (6), (4), and (3), we obtain that every automorphic solution of equation (1) admits the representation

$$U(z) = \exp[\omega(z) + \Gamma(z)]\psi(z). \tag{7}$$

This is precisely the analogue of representation (3) for automorphic generalized analytic functions. The function $\exp\{\omega(z) + \Gamma(z)\}$ is piecewise continuous on E with jump line L, does not vanish at any point of the plane E, is automorphic with respect to the group S, is continuous everywhere in R, and is continuously extendable to L.

From the above considerations one can draw the following conclusions:

1. In the fundamental domain R, the automorphic generalized analytic function $U(z)$ has the same discontinuities, zeros, and poles as the analytic automorphic function $\psi(z)$. If, in particular, $U(z)$ is continuous and bounded everywhere on R, then $\psi(z) \equiv \text{const} = c$ (4), and

$$U(z) = \exp[\omega(z) + \Gamma(z)]c. \tag{8}$$

Solutions of equation (1) represented in the form (8) are naturally (1) called **generalized constants on the fundamental polygon**.

2. A generalized analytic automorphic function, not identically equal to zero, has in the fundamental domain one and the same number of zeros and poles.

3. If the automorphic function $\psi(z)$ is piecewise meromorphic in R and is a solution of the homogeneous Riemann boundary-value problem (3)

$$\psi^+(t) = G_*(t)\psi^-(t), \qquad t \in L_0, \tag{9}$$

where L_0 is a system of a finite number of piecewise smooth closed or open curves located in R, then the corresponding generalized automorphic analytic function $U(z)$ will satisfy the same boundary condition

$$U^+(t) = G_*(t)U^-(t), \qquad t \in L_0, \tag{10}$$

and the conclusions on solvability and on the number of linearly independent solutions of problem (8) carry over without change to problem (10).

The uniformization theorem makes it possible to represent (5) every closed Riemann surface R^* of finite genus ρ by the fundamental polygon R of some Fuchsian group S of fractional-linear transformations. Thus, single-valued analytic functions on R^* become automorphic on E, and conversely. In this sense, the conclusions given above about generalized analytic automorphic functions carry over to single-valued generalized analytic functions on a closed Riemann surface. Obviously, a number of propositions obtained by Yu. L. Rodin in works (6–9), devoted to the study of generalized analytic functions on a closed Riemann surface, can be confirmed by starting from representations (7).

A representation analogous to (7) for a finite domain, on surfaces more general than a Riemann surface, was obtained by I. I. Danilyuk (^{10}).

Differential equation (1) is reduced (^{8}) to the integral equation

$$U(z) + \frac{1}{\pi} \iint\limits_{R} A(t)\overline{U}(t)H(z,t)f_1'(t)\,dT = \psi(z). \tag{11}$$

From representations (7) and (8) the unsolvability of the homogeneous ($\psi(z) \equiv 0$) equation (11) follows easily. On the basis of other considerations this fact was also proved by A. I. Serbin. The operator

$$-\frac{1}{\pi} \iint_{R} A(t)\,\overline{U}(t)\,H(z,t)\,f_1'(t)\,dT$$

is completely continuous $^{(8)}$; therefore, applying the method of successive approximations, we obtain

$$\exp\{\omega(z) + \Gamma(z)\} = 1 + \iint_{R} \overset{\psi}{\Gamma}_1(z,t)\,dT + \iint_{R} \overset{\psi}{\Gamma}_2(z,t)\,dT,$$

where $\overset{\psi}{\Gamma}_1$ and $\overset{\psi}{\Gamma}_2$ are the resolvents of the equation

$$\frac{\partial V}{\partial \overline{z}} = A\,\frac{\overline{\psi}}{\psi}\,\overline{V}.$$

Thus, representation (7) admits the following inversion:

$$U(z) = \psi(z)\left\{1 + \iint_{R} \overset{\psi}{\Gamma}_1(z,t)\,dT + \iint_{R} \overset{\psi}{\Gamma}_2(z,t)\,dT\right\}.$$

This formula makes it possible, for each automorphic analytic function $\psi(z)$ in R, to construct the corresponding generalized automorphic analytic function $U(z)$, and, evidently, by the scheme of L. G. Mikhailov $^{(11)}$ it makes it possible to solve, inside the fundamental domain R, the nonhomogeneous Riemann problem for generalized automorphic analytic functions

$$U^+(t) = G_*(t)\,U^-(t) + g_*(t), \qquad t \in L_0,$$

which is investigated by various methods by A. I. Serbin and Yu. L. Rodin $^{(6–8)}$.

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Note: Figure translations are in progress. See original paper for figures.

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