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## Abstract

## Full Text

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# RADIATION OF A POINT CHARGED PARTICLE FLYING ALONG THE AXIS OF A SEMI-INFINITE CIRCULAR WAVEGUIDE

*(Presented by Academician M. A. Leontovich on 25 I 1964)*

**1.** Consider a circular waveguide of radius $a$ with ideally conducting, infinitely thin walls. The waveguide is open at one end. For consideration of the problem it is convenient to use a cylindrical coordinate system $r, \varphi, z$, aligning the $z$ axis with the axis of the waveguide. We shall assume that the position of the waveguide walls is determined by the equations $r = a$, $z > 0$.

Let a point particle of charge $q$ move along the axis of the waveguide with velocity $u$. The problem is to determine the radiation arising when the particle enters the waveguide ($u > 0$) or when it exits from the waveguide ($u < 0$).

**2.** We shall describe the field excited by the charge by the Hertz vector $\vec{\Pi}$. Let us represent $\vec{\Pi}$ in the form of a sum

$$\vec{\Pi} = \vec{\Pi}^0 + \vec{\Pi}^1, \tag{1}$$

where $\vec{\Pi}^0$ describes the field of a point charge moving in empty space, and $\vec{\Pi}^1$ describes the free field, which is to be determined with the aid of boundary conditions.* The function $\vec{\Pi}^0$ is determined as the solution of the inhomogeneous d' Alembert equation and is written in the form

$$\Pi^0 = \frac{iq}{\pi} \int_{-\infty}^{\infty} e^{i\frac{\omega}{u}(z-ut)} K_0(k\gamma r) \frac{d\omega}{\omega}, \tag{2}$$

where the following notation has been introduced:

$$\gamma = \frac{\sqrt{1-\beta^2}}{|\beta|}, \qquad \beta = \frac{u}{c}, \qquad k = \frac{|\omega|}{c}. \tag{3}$$

The free field, determined by the function $\Pi^1$, is sought in the form

$$\Pi^1 = \int_{-\infty}^{\infty} \Pi^1_\omega e^{-i\omega t}\, d\omega. \tag{4}$$

In what follows we shall omit the index $\omega$ on the function $\Pi^1_\omega(r,z)$. It is convenient to represent the function $\Pi^1$ in the form of an expansion in a Fourier integral with respect to the variable $z$, writing this expansion in the following way:

$$\Pi^1(r,z) = -\frac{2\pi^2 a}{\omega} \int_{-\infty}^{\infty} F(w) \left\{ \begin{array}{c} J_0(va)\, H_0^{(1)}(vr) \\ J_0(vr)\, H_0^{(1)}(va) \end{array} \right\} e^{iwz}\, dw, \tag{5}$$

where $v = \sqrt{k^2 - w^2}$ $(\operatorname{Im} v \geq 0)$. For $r > a$ in the expression for $\Pi^1$ one should take the upper line in the braces under the integral sign, and for $r < a$ the lower line. The function $F(w)$, which remains to be determined, has the meaning of the Fourier component of the current induced by the moving charged par-

* The symmetry of the problem makes it possible to choose both vectors directed along the $z$ axis.

···flowing on the walls of the semi-infinite waveguide:

$$j(z) = \int_{-\infty}^{\infty} F(w) e^{iwz}\, dw. \tag{6}$$

As the boundary conditions of the problem we shall take the vanishing of the current density $j(z)$ on the extension of the waveguide walls $(r = a,\ z < 0)$ and the vanishing of the tangential component of the total electric field on the waveguide walls $(r = a,\ z > 0)$. These two conditions lead to the following system of paired integral equations for the function $F(w)$:

$$\int_{-\infty}^{\infty} F(w) e^{iwz}\, dw = 0 \qquad \text{for } z < 0,$$

$$\int_{-\infty}^{\infty} F(w) L(w) e^{iwz}\, dw = -\frac{iqk^2\gamma^2}{2\pi^2} K_0(k\gamma a) e^{i\frac{\omega}{u}z} \qquad \text{for } z > 0, \tag{7}$$

where

$$L(w) = \pi a v^2 J_0(va) H_0^{(1)}(va). \tag{8}$$

3. We shall solve the system of equations (7) with kernel (8) by the Wiener-Hopf method, analogously to how this was done by L. A. Vainshtein ([1]) for the case of diffraction of electromagnetic waves at the open end of a waveguide. We factor the kernel $L(w)$:

$$L(w) = v\varphi_1(w)\varphi_2(w), \tag{9}$$

where the function $\varphi_1(w)$ is analytic and has no zeros in the upper half-plane of the complex variable $w$, while the function $\varphi_2(w)$ has the same properties in the lower half-plane $w$. The explicit form of the functions $\varphi_1$ and $\varphi_2$ is given in ([1,2]). They are connected by the relation $\varphi_1(w) = \varphi_2(-w)$. The solution of the system of equations (7) has the form

$$F(w) = -\frac{q}{8\pi^3 ai I_0(k\gamma a)} \frac{\sqrt{k-\omega/u}\,\varphi_2(\omega/u)}{\sqrt{k-w}\,\varphi_2(w)} \frac{1}{w-\omega/u}. \tag{10}$$

Here the pole $w = \omega/u$ should be regarded as lying above the real axis.

4. Let us now consider the radiation field inside the waveguide. Substitution of (10) into (5) gives

$$\Pi^1 = \frac{q\sqrt{k-\omega/u}\,\varphi_2(\omega/u)}{4\pi i\omega I_0(k\gamma a)} \int_{-\infty}^{\infty} \frac{J_0(vr)H_0^{(1)}(va)e^{iwz}}{\sqrt{k-w}(w-\omega/u)\varphi_2(w)}\,dw. \tag{11}$$

To compute the field inside the waveguide ($z > 0$, $r < a$), it is important to know the behavior of the integrand in the upper half-plane of the complex variable $w$. The only singularities of the integrand in the upper half-plane are the poles located at the zeros of the function $\varphi_2(w)$ and at the point $w = \omega/u$. The integral (11) is readily evaluated by means of the residue theorem, which gives

$$\Pi^1(r,z) = \frac{q}{\pi i\omega} \frac{K_0(k\gamma a)}{I_0(k\gamma a)} I_0(k\gamma r)e^{i\frac{\omega}{u}z} +$$

$$+\frac{q\sqrt{k-\omega/u}\,\varphi_2(\omega/u)}{2\omega I_0(k\gamma a)} \sum_m \frac{J_0(v_m r)H_0^{(1)}(v_m a)e^{iw_m z}}{\sqrt{k-w_m}\,\varphi_2'(w_m)(w_m-\omega/u)}. \tag{12}$$

The first term in (12), together with $\Pi^0$, gives the field of a charge moving along the axis of an infinite circular waveguide. The sum over $m$ gives the radiation into the waveguide caused by the entry or exit of the charge. The spectral density

of radiation into the waveguide at frequency $\omega$ is

$$W_\omega = \frac{q^2}{\pi\omega} \frac{|k-\omega/u|\,|\varphi_2(\omega/u)|^2}{I_0^2(k\gamma a)} \sum_m \frac{\mathfrak{w}_m}{(k-\mathfrak{w}_m)\,|\varphi_2'(\mathfrak{w}_m)|^2(\mathfrak{w}_m-\omega/u)^2}. \tag{13}$$

The summation is carried out over those waves for which the longitudinal wave number $\mathfrak{w}_m$ is real.

5. Let us consider the radiation field outside the waveguide. We pass to spherical coordinates $z = R\cos\theta$, $r = R\sin\theta$. Evaluating expression (11) by the saddle-point method for large values of $R$, we obtain

$$\Pi^1 = -\frac{q}{2\pi\omega I_0(k\gamma a)}\frac{e^{ikR}}{R}\frac{\varphi_2(\omega/u)}{\varphi_2(k\cos\theta)}\frac{\sqrt{k-\omega/u}}{\sqrt{k-k\cos\theta}}\frac{J_0(ka\sin\theta)}{k\cos\theta-\omega/u}. \tag{14}$$

The intensity of radiation at frequency $\omega$ into the solid angle $d\Omega$ is

$$W_\omega(\theta)\,d\Omega = \frac{q^2|u|(1-\beta)\,|\varphi_2(\omega/u)|^2}{4\pi^2c^2I_0^2(k\gamma a)}\frac{J_0^2(ka\sin\theta)\sin^2\theta\,d\Omega}{(1-\beta\cos\theta)^2(1-\cos\theta)|\varphi_2(k\cos\theta)|^2}. \tag{15}$$

For small charge velocities the radiation spectrum lies in the region of low frequencies satisfying the inequality

$$k\gamma a = \frac{\omega}{u}\,a < 1. \tag{16}$$

For higher frequencies the radiation intensity decreases exponentially.

For large charge velocities ($\beta \simeq 1,\ \gamma \to 0$), the radiation spectrum lies in the region of frequencies satisfying the inequality

$$\omega < \frac{u}{\sqrt{1-\beta^2}\,a} = \frac{c}{\gamma a}. \tag{17}$$

For high frequencies one may use the circumstance that the function $\varphi_2$, for large values of its argument, tends to unity. For definiteness let us consider the case of the charge emerging from the waveguide ($u < 0$) and take into account that the radiation of a fast charge is concentrated in the range of angles $\pi-\theta \simeq \gamma$. In this case

$$W_\omega(\theta) = \frac{q^2}{4\pi^2c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}. \tag{18}$$

The expression obtained coincides with the expression for the intensity of transition radiation when a charge emerges from a plane metal boundary [3]. The expression for the radiation losses when a charge passes along the axis of a circular aperture in an ideally conducting plane screen [4,5] is also brought to an analogous form.

An estimate of the total losses to radiation into free space when a relativistic charge emerges from the waveguide gives

$$W = \frac{2q^2}{\pi\gamma a}. \qquad (19)$$

The geometrical-optics approximation, which is applicable as $\gamma \to 0$, leads to the same qualitative results.

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*Note: Figure translations are in progress. See original paper for figures.*