CYBERNETICS AND CONTROL THEORY

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Abstract

Full Text

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ON A PROBLEM IN THE SYNTHESIS OF CONTROLLED SYSTEMS

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A system of automatic control is considered whose behavior under the influence of disturbing actions is described on the time interval $[0, T]$ by a linear differential equation of order n,

$$L(y) = f(t) \tag{1}$$

with zero initial conditions. The disturbance $f(t)$ belongs to the class F of piecewise-continuous functions bounded in absolute value by the constant m_0.

B. V. Bulgakov ([1]) solved the problem of determining a function $f^*(t)$ from F for which the corresponding solution attains the greatest absolute value at the time T (the maximum accumulated error). This problem was also considered in ([2,3]).

The present communication is devoted to the problem of synthesis of automatic control systems according to the principle of minimizing the maximum accumulated error. The case studied is when

$$L(y) = L_1(y) + c(t)y.$$

Here $L_1(y)$ is an operator of order n with constant or variable coefficients, corresponding to the unchangeable part of the system. The coefficient $c(t)$, which is to be chosen, belongs to the class H of piecewise-continuous functions bounded in absolute value by the constant m_1, and has the physical meaning of a variable gain coefficient.

Denote the solution of equation (1) by $y(t, f(t), c(t))$. It is required to find such a $c^0(t)$ from H at which

$$I = \min_c \max_f |y(T, f(t), c(t))|, \qquad c \in H, \quad f \in F, \quad 0 \le t \le T.$$

A method is presented that makes it possible to determine successively functions $c_i^0(t)$ from H in such a way that

$$R_{i+1} = \max_f |y(T, f(t), c_{i+1}^0(t))| < R_i = \max_f |y(T, f(t), c_i^0(t))|.$$

Sufficient conditions are indicated under which this process leads to the determination of I; an estimate is given for the number of steps required for this, and an estimate of the number of switchings in the function $c^0(t)$. The results obtained can also be extended to the case where it is necessary to choose the coefficients at the derivatives of $y(t)$ of order less than $n-1$.

Let $y_1(t, f)$ be the solution of the equation $L_1(y) = f$ with zero initial conditions; $G_1(T, t)$ the Cauchy function of the operator $L_1(y)$. Then $f_1(t) = m_0 \operatorname{sign} G_1(T, t)$ is the function on which the greatest possible value of the absolute value of the solution at the time T is attained. Choose an arbitrary point t_0 from $[0, T]$ for which $y_1(t_0, f_1) \neq 0$. Put

$$c(t) = m_1 \operatorname{sign}(y_1(t_0, f_1) G_1(T, t)) \quad \text{for } t \in \gamma, \qquad c(t) = 0 \quad \text{for } t \in \bar{\gamma};$$

$$\gamma = [t_0 - \mu, t_0 + \nu]; \quad \mu \geq 0, \quad \nu \geq 0, \quad \nu - \mu = \Delta, \quad \gamma \in [0, T]. \tag{2}$$

Let $c(t)$ in (1) be defined by relations (2), and let $y(t, f, \Delta)$ be the solution of equation (1) with zero initial conditions. By $f_\alpha(t)$ denote such a function from F on which

$$\max_f((y_1(t_0, f_1) - y_1(t_0, f)) \operatorname{sign} y_1(t_0, f_1)) \quad \text{for } y_1(T, f_1) - y_1(T, f) = \alpha$$

$$0 \leq \alpha < 2y_1(T, f_1), \qquad f \in F.$$

This function, for $y_1(t_0, f_1) > 0$, has the form

$$f_\alpha(t) = -m_0 \operatorname{sign} G_1(T, t) \quad \text{for } t \in \sigma(x_\alpha),$$

$$f_\alpha(t) = m_0 \operatorname{sign} G_1(T, t) \quad \text{for } t \notin \sigma(x_\alpha).$$

Here $\sigma(x)$ is the set in $[0, T]$ on which $G^*(t) \geq x$;

$$G^*(t) = \frac{G_1(t_0, t)}{G_1(T, t)} \quad \text{for } t \in [0, t_0]; \qquad G^*(t) = 0 \quad \text{for } t \in (t_0, T].$$

By x_α we denote the unique root of the equation

$$m_0 \int_{\sigma(x)} |G_1(T,\tau)|\, d\tau = \frac{\alpha}{2}.$$

If

$$m_0 \int_{\sigma(+0)} |G_1(T,\tau)|\, d\tau < \frac{\alpha}{2}, \qquad m_0 \int_{\sigma(-0)} |G_1(T,\tau)|\, d\tau > \frac{\alpha}{2},$$

then

$$f_\alpha(t) = -m_0 \operatorname{sign} G_1(T,t) \quad \text{for } t \in \sigma(+0) \cup (t_0, u_0),$$

$$f_\alpha(t) = m_0 \operatorname{sign} G_1(T,t) \quad \text{for } t \notin \sigma(+0) \cup (t_0, u_0).$$

Here u_0 is the unique root of the equation

$$m_0 \int_{\sigma(+0)} |G_1(T,\tau)|\, d\tau + m_0 \int_{t_0}^{u} |G_1(T,\tau)|\, d\tau = \frac{\alpha}{2}.$$

If $G^*(t) = \text{const}$ for $t \in [0, t_0)$, then

$$f_\alpha(t) = -m_0 \operatorname{sign} G_1(T,t) \quad \text{for } t \in [0, u_1],$$

$$f_\alpha(t) = m_0 \operatorname{sign} G_1(T,t) \quad \text{for } t \in (u_1, T].$$

Here u_1 is the root of the equation

$$m_0 \int_0^u |G_1(T,\tau)|\, d\tau = \frac{\alpha}{2}.$$

If $y_1(t_0, f_1) < 0$, then $f_\alpha(t)$ is obtained by a slight modification of the formulas (4) given above. The expression $y_1(t_0, f_\alpha)$ is a convex function of the argument α, vanishing at the point α_0. Let us also denote

$$a_0 = T \sup_{\tau,t} \left| \frac{\partial G_1(t,\tau)}{\partial t} \right| + \sup_{\tau,t} |G_1(t,\tau)|, \qquad a_1 = \sup_{\tau,t} |G_1(t,\tau)|$$

$$(0 \le \tau \le t \le T).$$

Theorem 1. *Suppose that the quantity* Δ *specified in (2) does not exceed* $\min\{T, \Delta^0\}$*; then for any* f *from* F

$$|y(T, f, \Delta)| \leq y_1(T, f_1) - \frac{\alpha_0 m_1 |y_1(t_0, f_1)|}{4 y_1(T, f_1)} \int_{t_0-\mu}^{t_0+\nu} |G_1(T, \tau)| \, d\tau;$$

$$\Delta^0 = \frac{4\alpha_0 y_1(T, f_1)}{\alpha_0 m_1 a_1 |y_1(t_0, f_1)| + 4q y_1(T, f_1)}, \tag{4}$$

$$q = \max\left\{4m_1 a_1 a_0 + 8\frac{m_0 a_0 y_1(T, f_1)}{|y_1(t_0, f_1)|},\ 2m_1 a_1 y_1(T, f_1)\right\},$$

We describe a multistage process of decreasing the maximum accumulated error. Define the differential operator L_{i+1} from the relations

$$L_{i+1} = L_i + c_i y \quad (i = 1, 2, ...), \qquad c_i^-(t) \leq c_i(t) \leq c_i^+(t),$$

$$c_i^+(t) = c_{i-1}^+(t) - c_{i-1}^*(t), \qquad c_i^-(t) = c_i^+(t) - 2m_1, \qquad c_1^+(t) = m_1.$$

Here $c_{i-1}^*(t)$ is the function chosen at the $(i-1)$-st step. Denote by E_i the subset of $[0, T]$ on which

$$\operatorname{sign} c_i^-(t) \leq \operatorname{sign}(y_i(t, f_i) G_i(T, t)) \leq \operatorname{sign} c_i^+(t) \quad (i = 2, 3, ...), \qquad E_1 = [0, T].$$

In these inequalities $y_i(t, f_i)$ is the solution of the equation $L_i(y) = f_i$ with zero initial conditions; $G_i(T, t)$ is the Cauchy function of the operator L_i,

$$f_i(t) = m_0 \operatorname{sign} G_i(T, t).$$

Suppose that $i-1$ steps have been carried out; $\operatorname{mes} E_i = e_i$, and the Cauchy function $G_i(T, t)$ has s_i zeros on $(0, T)$. The set E_i contains either an interval E_{i1} of length greater than $e_i/2s_i$, or a subset E_{i1} of measure greater than $e_i/2$, which consists only of intervals whose boundary points are zeros of $y_i(t, f_i)$ and discontinuity points of the function $c_i^+(t)$.

On E_{i1} we choose a point t_i at which $y_i(t, f_i)$ attains its greatest value in absolute value. The subinterval of E_{i1} on which t_i lies is denoted by β_{i1}. From relations (3), (4), applied to the operator L_i and the point t_i, we determine Δ_i^0 (if $m_1^* = \max\{|c_i^+(t_i)|, |c_i^-(t_i)|\} = 2m_1$, then in (3), (4) $2m_1$ must be substituted instead of m_1). Let γ_i denote the interval $[t_i - \mu_i, t_i + \nu_i]$ containing the point t_i. If $\operatorname{mes} \beta_{i1} \leq \Delta_i^0$, then γ_i coincides with β_{i1}. If $\operatorname{mes} \beta_{i1} > \Delta_i^0$, then $\gamma_i \subset \beta_{i1}$, and $\operatorname{mes} \gamma_i = \Delta_i^0$. We define the function $c_i^*(t)$ by the relations:

$$c_i^*(t) = m_1^* \operatorname{sign}(y_i(t_i, f_i) G_i(T, t)) \quad \text{for } t \in \gamma_i, \qquad c_i^*(t) = 0 \quad \text{for } t \notin \gamma_i.$$

In order that I be realized at the $(i-1)$-st step, it is necessary that

$$\operatorname{mes} E_i = 0. \tag{5}$$

Some sufficient conditions are given by

Theorem 2. *Suppose that after the $(i-1)$-st step condition (5) is satisfied. If, moreover: 1) $y_i(\tau, f_i) > 0$ for every τ in $(0, T)$; 2) $G_i(t, \tau)$, considered for fixed τ as a function of the argument t, is sign-constant for $\tau \le t \le T$, then $y_i(T, f_i) = I$.*

An estimate of the degree of decrease of the maximal accumulated error R_i and of the measure e_i of the set E_i as functions of the number of steps is given by

Theorem 3. *Let A, ε be arbitrary positive numbers satisfying the inequalities $A < y_1(T, f_1)$, $\varepsilon < T$. Then after the r-th step either $y_{r+1}(T, f_{r+1}) < A$, or $\operatorname{mes} E_{r+1} < \varepsilon$. Here*

$$r \le r^0 = 3\left[\frac{y_1(T, f_1) - A}{\lambda} + 1\right], \qquad \lambda = k A^{s_1} \varepsilon^{s_2}.$$

The function $c^0(t)$, on which the minimum of the maximal accumulated error I is realized, can take only two values: m_1 and $-m_1$. An estimate of the number of switching points of $c^0(t)$ on the interval $(0, T)$ is given by

Theorem 4. *If $I > A > 0$, then the function $c^0(t)$ has on $(0, T)$ no more than $\eta = [k_1/A]$ switching points.*

In Theorems 3 and 4 the square brackets denote the integer part of a number. The coefficients k, k_1 depend on the upper bound of the modulus of the coefficients of the operator $L_1(y)$ and of the adjoint operator $L_1^*(y)$, on the order of equation (1), and on the quantities m_1, m_0, T. The coefficients s_1 and s_2 are determined by the order of equation (1).

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Note: Figure translations are in progress. See original paper for figures.