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Abstract

Full Text

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ON THE CARDINALITY OF SETS OF CLOSED CLASSES OF FINITE HEIGHT IN $P_{\aleph_0}$

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Following ([1]), denote by $P_\mathfrak{m}$ the set of all functions whose domains of definition and ranges of values are the set $E^\mathfrak{m}$ of cardinality $\mathfrak{m}$ ($\mathfrak{m} \geq 2$). Let Σ be an operation assigning to each subset $M \subseteq P_\mathfrak{m}$ a subset $\Sigma(M) \subseteq P_\mathfrak{m}$. We assume that the operation Σ satisfies the conditions: 1) $M \subseteq \Sigma(M)$; 2) $\Sigma(\Sigma(M)) = \Sigma(M)$; 3) if $M_1 \subseteq M$, then $\Sigma(M_1) \subseteq \Sigma(M_2)$. A set $M \subseteq P_\mathfrak{m}$ is called a closed class (with respect to the operation Σ) if $\Sigma(M) = M$. Consider the partially ordered set of all closed classes in $P_\mathfrak{m}$ with the order relation being the usual set-theoretic inclusion. We shall call this set the inclusion structure of the set $P_\mathfrak{m}$.

Definition (inductive). $P_\mathfrak{m}$ is a class of height 0. Let M be an arbitrary closed class of functions from $P_\mathfrak{m}$. We shall call it a class of height l (l a positive integer) if, whatever function f from $P_\mathfrak{m} \setminus M$ may be taken, the class $\Sigma(M \cup \{f\})$ has height $n \leq l-1$, and there exists a function f_0 from $P_\mathfrak{m} \setminus M$ such that the height of the class $\Sigma(M \cup \{f_0\})$ is equal to $l-1$. If, however, for any finite l there are functions $f_1, \dots, f_l$ from $P_\mathfrak{m}$ such that: 1) $f_1 \notin M$; 2) for every i ($1 \leq i \leq l-1$), $f_{i+1} \notin \Sigma(M \cup \{f_1, \dots, f_i\})$; 3) $(M \cup \{f_1, \dots, f_l\}) \neq P_\mathfrak{m}$, then the set M will be called a class of infinite height.

By $K_\mathfrak{m}^{(l)}$ ($1 \leq l < \aleph_0$) we shall denote the set of all classes of height l in the set $P_\mathfrak{m}$.

In solving problems connected with the functional systems $P_\mathfrak{m}$, it is often necessary to obtain various information about the structure of the inclusion structures of the sets $P_\mathfrak{m}$. In what follows the operation Σ is the operation of superposition ([2]). E. Post ([3]) described in detail the inclusion structure of the algebra of logic (P_2). Analyzing it, one can establish a number of interesting properties of the system P_2: obtain answers to questions on the completeness of various subsets of P_2, find bases of particular closed classes, etc. However, already in three-valued logic (P_3) attempts to construct the inclusion structure are associated with great difficulties of a purely fundamental character. The nature of these difficulties is partly revealed in the work ([4]) of A. A. Muchnik and Yu. I. Yanov: among other results, it was established there that the cardinality of the set of closed classes in $P_\mathfrak{m}$ ($3 \leq \mathfrak{m} < \aleph_0$) is equal to the cardinality of the continuum ($\mathfrak{c}$). But, on the other hand, statements are known which show that an uncountable branching of the inclusion structure of the systems $P_\mathfrak{m}$ ($3 \leq \mathfrak{m} < \aleph_0$)

occurs in the set of classes of sufficiently large height. Such statements include, first of all, the theorem of A. V. Kuznetsov (^{2}) on the finiteness of the set of all precomplete classes (classes of height 1) in $P_{\mathfrak{m}}$ ($\mathfrak{m} < \aleph_0$). Also interesting in this respect is the work of V. M. Gnidenko (^{5}): from it there easily follows the finiteness of the set $K_3^{(2)}$.

Let us note that in $P_{\mathfrak{m}}$ ($3 \leq \mathfrak{m} < \aleph_0$), apart from the facts indicated, there are as yet no results concerning the cardinalities of the sets $K_{\mathfrak{m}}^{(l)}$. In countably valued logic ($P_{\aleph_0}$), as S. V. Yablonskii showed in 1959, the cardinality of the set of all closed classes is equal to the cardinality of the hypercontinuum (2^c). The substantially greater complexity of the inclusion structure of the set $P_{\aleph_0}$ was also indicated by the previously known fact (^{1}) that the cardinality of the set $K_{\aleph_0}^{(1)}$ is not less than c. In connection with this, in $P_{\aleph_0}$ such characteristics of the inclusion structure acquire still greater value as make it possible to estimate, sufficiently accurately, the cardinalities of various sets of closed classes.

In the present note we give a brief exposition of the proof of the assertion on the hypercontinuity of the set $K_{\aleph_0}^{(1)}$. This result shows that every completeness criterion in countably valued logic which uses an "essential part" of the set of all precomplete classes is equivalent, "in the cardinal sense," to an enumeration of the set of all closed classes in $P_{\aleph_0}$. We note that a minor modification of the construction built by us for proving the hypercontinuity of the set $K_{\aleph_0}^{(1)}$ makes it possible to establish the following assertion: the cardinality of the set $K_m^{(1)}$ for $m \geq \aleph_0$ is equal to 2^{2^m}. In addition, the note formulates an assertion from which follows the hypercontinuity of the sets $K_{\aleph_0}^{(l)}$ for $l = 2, 3, \ldots$.

In proving the existence of a hypercontinuous set of precomplete classes we use the following theorem (it is easily established, for example, with the aid of F. Hausdorff' s theorem (^{6}) and the fact that (^{7}) any countable set of functions from $P_{\aleph_0}$ can be embedded in a closed class generated by one function).

Theorem 1. *Let $\mathfrak{M}$ be a closed class distinct from the entire set $P_{\aleph_0}$. Suppose, moreover, that there exists an at most countable set $\mathfrak{N}$ of functions from $P_{\aleph_0}$ which, together with the set $\mathfrak{M}$, forms a complete system in $P_{\aleph_0}$. Then the class $\mathfrak{M}$ can be extended to a precomplete class in $P_{\aleph_0}$.*

1°. Define the functions $\varphi_n(x_0, x_1, \ldots, x_n)$, $n \geq 0$:

$$\varphi_0(x_0) = x_0, \quad \varphi_1(x_0, x_1) = \frac{(x_0 + x_1)(x_0 + x_1 + 1)}{2} + x_1, \ldots$$

$$\ldots, \varphi_n(x_0, x_1, \ldots, x_{n-1}, x_n) = \varphi_1(\varphi_{n-1}(x_0, x_1, \ldots, x_{n-1}), x_n), \ldots$$

By induction on the number n it is easily proved:

Lemma 1. *The function φ_n maps the set*

$$\underbrace{E^{\aleph_0} \times \cdots \times E^{\aleph_0}}_{n+1 \text{ times}}$$

one-to-one onto the set $E^{\aleph_0}$.

Partition the set $E^{\aleph_0}$ into $\aleph_0$ infinite subsets:

$$E^{\aleph_0} = \bigcup_{p=0}^{\infty} \mathcal{E}_p, \quad \mathcal{E}_p = \{e_0^{(p)}, e_1^{(p)}, \dots, e_q^{(p)}, \dots\}; \quad \mathcal{E}_{p_1} \cap \mathcal{E}_{p_2} = \emptyset \text{ for } p_1 \neq p_2.$$

Next partition each of the sets $\mathcal{E}_p$ in the following way:

$$\mathcal{E}_p = \{e_0^{(p)}\} \cup \bigcup_{r=0}^{\infty} \mathcal{E}_r^{(p)}, \quad \text{where } \mathcal{E}_r^{(p)} = \{e_{2^r}^{(p)}, e_{2^r\cdot 3}^{(p)}, \dots, e_{2^r\cdot(2s+1)}^{(p)}, \dots\}.$$

Denote by $\mathfrak{A}$ the set of all infinite sequences of nonnegative integers. Let $A = \{a_0, a_1, \dots, a_k, \dots\}$ be an arbitrary sequence from the set $\mathfrak{A}$. Introduce the function $\alpha(m, l, A)$:

$$\alpha(m, l, A) = 2^{\varphi_m(a_0, a_1, \dots, a_m)}(2l + 1).$$

Using this function, define, for any sequence A from the set $\mathfrak{A}$, a function $g_A(x)$:

$$g_A(x) = e_{\alpha(p,q,A)}^{(p)}, \quad \text{if } x = e_q^{(p)} \quad (p = 0, 1, \dots;\ q = 0, 1, \dots).$$

It is easy to see (see also Lemma 4) that if A_1 and A_2 are two distinct sequences from $\mathfrak{A}$, then $g_{A_1}(x) \neq g_{A_2}(x)$. Hence, incidentally, it follows that the set $\{g_A(x) : A \in \mathfrak{A}\}$ has cardinality continuum.

Let $A_k = \{a_{k0}, a_{k1}, \dots, a_{km}, \dots\}$ and $A'_l = \{a'_{l0}, a'_{l1}, \dots, a'_{ln}, \dots\}$, where $k = 1, \dots, i;\ l = 1, \dots, j;\ i \geq 1,\ j \geq 1$.

Lemma 2. If the function $g_{A_1}(g_{A_2} \cdots g_{A_i}(x) \cdots)$ coincides with the function $g_{A'_1}(g_{A'_2} \cdots g_{A'_j}(x) \cdots)$ for some $x \in \mathcal{E}_p$, then $i = j$ and $a_{km} = a'_{km}$, $m = 0, 1, \dots, p;\ k = 1, 2, \dots, i$.

Corollary. If the condition of Lemma 2 is fulfilled, then the functions $g_{A_1}(g_{A_2} \cdots g_{A_i}(x) \cdots)$ and $g_{A'_1}(g_{A'_2} \cdots g_{A'_j}^{(x)})$ coincide for every x from the set $\bigcup_{t=0}^{p} \mathcal{E}_t$.

Lemma 3. The equality

$$g_{A_1}(g_{A_2} \cdots g_{A_i}(x) \cdots) = x \qquad (i \geq 1)$$

does not hold for any value of the variable x.

Let $\mathfrak{A}'$ be an arbitrary nonempty proper subset of $\mathfrak{A}$, different from all of $\mathfrak{A}$.

From Lemma 2 it follows:

Lemma 4. Whatever the sequences $A_1, \dots, A_i, A'_1, \dots, A'_j$ $(i \geq 0,\ j \geq 1)$ from the set $\mathfrak{A}'$ and A_0 from the set $\mathfrak{A} \setminus \mathfrak{A}'$ may be, there exists a number $p_0 \geq 0$ such that the equality

$$g_{A_0}(g_{A_1}(g_{A_2} \cdots g_{A_i}(x) \cdots)) = g_{A'_1}(g_{A'_2} \cdots g_{A'_j}(x) \cdots)$$

is not satisfied for any $x \in \mathcal{E}_p$, provided only that $p \geq p_0$; moreover, if there exists an x^0 such that the indicated equality is true for $x = x^0$, then $i + 1 = j$.

Put, as usual, $\overline{\operatorname{sg}}\, x = 1$ for $x = 0$; $\overline{\operatorname{sg}}\, x = 0$ for $x \neq 0$.

Let $g_0(x)$ be some function mapping the set $E^{\aleph_0}$ one-to-one onto the set $\mathcal{E}_0$.

Define the set $\mathfrak{C}_{\mathfrak{A}'}$ $(\mathfrak{C}_{\mathfrak{A}'} \subset P_{\aleph_0})$:

1) $\mathfrak{C}_{\mathfrak{A}'}$ contains any function $g_A(x)$ such that $A \in \mathfrak{A}'$;
2) $\mathfrak{C}_{\mathfrak{A}'}$ contains all functions $\chi_{B,g}(x, y) \cdot \overline{\operatorname{sg}}\, |g_B(x) - y|$, where B ranges over the set $\mathfrak{A} \setminus \mathfrak{A}'$, and $g(x)$ ranges over the set of all functions in $P_{\aleph_0}$ depending on one variable;
3) $\mathfrak{C}_{\mathfrak{A}'}$ contains the function $f_0(x, y)$:

$$f_0(x, y) = \begin{cases} g_0(\varphi_1(x, y)), & \text{if } (x, y) \in \mathcal{E}_0 \times \mathcal{E}_0, \\ 0, & \text{if } (x, y) \notin \mathcal{E}_0 \times \mathcal{E}_0, \end{cases}$$

   where $\varphi_1(x, y) = (x + y)(x + y + 1)/2 + y$;
4) $\mathfrak{C}_{\mathfrak{A}'}$ contains no other functions except those listed in items 1), 2), and 3).

By $[\mathfrak{M}]$ we shall denote the closure (with respect to the operation of superposition) of the set $\mathfrak{M}$.

Lemma 5. If $h_0(x) \in [\mathfrak{C}_{\mathfrak{A}'}]$, then: a) either $h_0(x)$ is a superposition of functions from the set $\{g_A(x) : A \in \mathfrak{A}'\}$; b) or there exists a number $p_0 \geq 0$ such that on the set $\bigcup\limits_{p \geq p_0} \mathcal{E}_p$ the function $h_0(x)$ assumes a constant value.

Corollary.

$$[\mathfrak{C}_{\mathfrak{A}'}] \cap \{g_A(x) : A \in \mathfrak{A}\} = \{g_A(x) : A \in \mathfrak{A}'\}.$$

Lemma 6. If $A \in \mathfrak{A} \setminus \mathfrak{A}'$, then

$$[\mathfrak{C}_{\mathfrak{A}'} \cup \{g_A(x)\}] = P_{\aleph_0}.$$

It follows immediately from Lemma 6 and Theorem 1 that

Lemma 7. *The class* $[\mathfrak{S}_{\mathfrak{A}'}]$ *admits an extension to a precomplete class in* $P_{\aleph_0}$, *namely to the class* $\mathfrak{K}_{\mathfrak{A}'}$.

Lemma 8. *If* $\mathfrak{A}'_1$ *and* $\mathfrak{A}'_2$ *are arbitrary nonempty subsets of* $\mathfrak{A}$ *such that* $\mathfrak{A}'_i \neq \mathfrak{A}$ $(i = 1, 2)$ *and* $\mathfrak{A}'_1 \neq \mathfrak{A}'_2$, *then* $\mathfrak{K}_{\mathfrak{A}'_1} \neq \mathfrak{K}_{\mathfrak{A}'_2}$.

It is obvious that the set of all subsets of countable-valued logic has cardinality 2^c. Taking into account the fact that the set of all subsets $\mathfrak{A}'$ of the set $\mathfrak{A}$ satisfying the conditions $\mathfrak{A}' \neq \emptyset$ and $\mathfrak{A} \setminus \mathfrak{A}' \neq \emptyset$ has cardinality 2^c, and using Lemmas 7 and 8, we obtain the following assertion.

Theorem 2. *The cardinality of the set of all precomplete classes in* $P_{\aleph_0}$ *is equal to the cardinality of the hypercontinuum.*

2°. We have established a stronger assertion than Theorem 2. Let us formulate it.

Theorem 3. *For every integer* l $(l \geq 0)$ *there exists in* $P_{\aleph_0}$ *a class of height* l, *containing* 2^c *classes of height* $l + 1$.

In the proof of this theorem, the classes of height l (for each $l \geq 1$) and the 2^c classes of height $l + 1$ contained in them are constructed by us directly.

A hypercontinuum family of classes in $P_{\aleph_0}$ having infinite height was constructed in 1959 by S. V. Yablonskii.

In conclusion, we note that the ideas of the constructions built by us for the proof of Theorem 3 arose in connection with the works of B. Neumann ([8]) (see also ([9]), pp. 245-248) and A. G. Kurosh ([10]).

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Note: Figure translations are in progress. See original paper for figures.