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Abstract

Full Text

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ON THE ORDEREDNESS OF CHARACTERISTIC MATRIX-FUNCTIONS OF DISSIPATIVE VOLTERRA OPERATORS

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Let us consider the class Ω_2^+ of all simple dissipative Volterra operators with two-dimensional imaginary component, acting in a separable Hilbert space $\mathfrak{H}$, and let K_2^+ be the set of the corresponding characteristic matrix-functions. By virtue of (1), the one-cell character of an operator $A \in \Omega_2^+$ is equivalent to the orderedness of its characteristic matrix-function $W(z) \in K_2^+$. In the present paper, conditions are established for the orderedness of matrix-functions from K_2^+.

1. Every matrix-function $W(z) \in K_2^+$ can be represented (2) in the form of a multiplicative Stieltjes integral

$$W(z) = \int_0^l e^{2iz\,dH(t)} \quad \left(\dot{H}(t) = \int_0^t P(x)\,dx, \quad P(x) \geq 0, \quad \operatorname{sp} P(x) = 1\right), \tag{1}$$

where it may be assumed (3) that the rank of the matrix-function $P(t)$ is equal to 1 almost everywhere. Representation (1) is unique if and only if the matrix-function $W(z)$ is ordered. It follows from (3) that every matrix-function $W(z) \in K_2^+$ is uniquely representable in canonical form

$$W(z) = e^{im_0 zI} \int_0^{l_0} e^{2iz\,dH_0(t)} = e^{im_0 zI} W_0(z), \quad \text{where} \quad H_0(t) = \int_0^t P_0(x)\,dx; \tag{2}$$

$P_0(x)$ is a Hermitian nonnegative matrix-function with summable elements, whose values are projection matrices of rank 1; $m_0 = 2l - \sigma$, where $2l$ is the weight and σ is the type of the matrix-function $W(z)$. Here the matrix-function $W_0(z)$ is equal to I for $\sigma = l$ and is ordered if $\sigma > l$. Since the scalar matrix-function $e^{im_0 zI}$ ($m_0 > 0$) is not ordered (3), for the matrix-function $W(z)$ to be ordered it is necessary and sufficient that it have no scalar divisors. We shall call an ordered divisor* of the matrix-function $W(z)$ maximal if its type is equal to the type σ of the matrix-function $W(z)$. In (4) the existence of maximal ordered divisors was established and it was proved that every such divisor has the form $W_3(z)W_0(z)$, where $W_3(z)$ is some matrix-function from K_2^+ of weight m_0. Hence follows

Lemma 1. If two maximal ordered divisors of $W(z)$ have no common divisors, then the matrix-function $W(z)$ is scalar.

Lemma 2. If

$$W_1(z) = \int_0^{l_1} e^{2iz\,dH_1(t)} \left(H_1(t) = \int_0^t P_1(x)\,dx \right)$$

is an ordered divisor of the matrix-function $W(z)$ and $2l_1 > m_0$, then $P_1(t) \equiv P_0(t)$ $(0 \leq t \leq l_1 - m_0/2)$.

Proof. We shall first prove that the matrix-functions $W_1(z)$ and $W_0(z)$ have a common divisor. Assuming the contrary, take a maximal ordered divisor of the matrix-function $W(z)$ and consider its divisor $W_2(z)$ of weight $2l_1$. Then, as is easy to show, the matrix-functions $W_1(z)$ and $W_2(z)$ also have no common divisors. Let the matrix-function $W(z)$ be characteristic for some operator $A \in \Omega_2^+$. There exist $(^2)$ invariant subspaces of the operator A relative to the operator A corres–

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* According to $(^2)$, a divisor of $W(z)$ is its right divisor.

spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, so that $W_k(z) = \mathrm{Pr}_{\mathfrak{H}_k} W(z)$ $(k = 1, 2)$. Denote by $\mathfrak{G}_0$ the smallest invariant subspace of the operator A containing $\mathfrak{H}_1$ and $\mathfrak{H}_2$, and let $V_0(z) = \mathrm{Pr}_{\mathfrak{G}_0} W(z)$. We shall show that the type σ_0 of the matrix function $V_0(z)$ is equal to $2l_1$. Indeed, since the matrix functions $W_k(z)$ $(k = 1, 2)$ are ordered and are divisors of $V_0(z)$, it follows that $\sigma_0 \geqslant 2l_1$. On the other hand, in $(^2)$ it was proved that $\sigma_0 \leqslant 2l_1$. Thus, $W_1(z)$ and $W_2(z)$ are maximal ordered divisors of the matrix function $V_0(z)$. Therefore, by Lemma 1, the matrix function $V_0(z)$ must be scalar, that is, $V_0(z) = e^{2il_1 z} I$, which is impossible, since $V_0(z)$ is a divisor of $W(z)$ and $2l_1 > m_0$. In view of the orderedness of the matrix functions $W_1(z)$ and $W_0(z)$, $P_1(t) \equiv P_0(t)$ on some interval of the form $[0, a]$ $(a > 0)$. Let $[0, a_0]$ be the largest interval on which the matrix functions $P_1(t)$ and $P_0(t)$ coincide. Then it is not difficult to prove that $a_0 \geqslant l_1 - m_0/2$. We note that an assertion analogous to Lemma 2 also holds for left divisors.

Lemma 3. Let the matrix functions

$$W_k(z) = \int_0^a e^{2iz\,dH_k(t)} \quad \left(H_k(t) = \int_0^t P_k(x)\,dx, \quad k = 1, 2 \right)$$

be ordered. If

$$W_2(z)W_1(z) = e^{2iaz} I,$$

then

$$P_2(t)P_1(a - t) \equiv 0.$$

Theorem 1. If the matrix functions

$$W_k(z) = \int_0^{l_k} e^{2iz\,dH_k(t)} \quad \left(H_k(t) = \int_0^t P_k(x)\,dx, \quad k = 1, 2, \dots, n\right)$$

are ordered, then, for the orderedness of the matrix function

$$W(z) = W_n(z) \cdots W_1(z),$$

it is necessary and sufficient that, for each $k = 1, \dots, n-1$, the measure of the set of all t from the interval $(0, \delta)$ for which

$$P_{k+1}(t) P_k(l_k - t) = 0$$

be positive for every $\delta > 0$.

Proof. It is sufficient to prove the theorem for $n = 2$. Suppose that the matrix function $W(z)$ is not ordered. Representing it in canonical form and using Lemma 2, we obtain

$$P_1(t) \equiv P_0(t) \quad (0 \leqslant t \leqslant l_1 - m_0/2);$$

$$P_2(l_2 - t) \equiv P_0(l_0 - t) \quad (0 \leqslant t \leqslant l_2 - m_0/2),$$

and since

$$(l_1 - m_0/2) + (l_2 - m_0/2) = l_1 + l_2 - m_0 = l_0,$$

it follows that

$$\int_0^{m_0/2} e^{2iz\,dH_2(t)} \times \int_{l_1 - m_0/2}^{l_1} e^{2iz\,dH_1(t)} = e^{im_0 z} I,$$

whence, by Lemma 3,

$$P_2(t) P_1(l_1 - t) \equiv 0 \quad (0 \leqslant t \leqslant m_0/2).$$

If

$$P_2(t) P_1(l_1 - t) \equiv 0 \quad (0 \leqslant t \leqslant \delta_0),$$

then, as is easy to show, the matrix function $W(z)$ has a scalar divisor and, consequently, is not ordered.

2. Lemma 4. Let

$$W(z) = \int_{x_0}^{x} e^{2iz\,dH(t)} \quad \left(H(t) = \int_{x_0}^{t} P(\xi)\,d\xi\right).$$

If

$$\sup_{t', t'' \in [x_0, x]} \|P(t') - P(t'')\| = \omega,$$

then

$$\|W(iy)\| \geqslant e^{-2y(1-\omega)\Delta x} \quad (\Delta x = x - x_0,\ y < 0).$$

Lemma 5. Let

$$W_k(z) = \int_0^l e^{2iz\,dH_k(t)} \quad \left(H_k(t) = \int_0^t P_k(x)\,dx, \quad k = 1, 2 \right).$$

If

$$mE(P_1 \neq P_2) \leqslant \varepsilon,$$

then

$$\|W_1(z) - W_2(z)\| \leqslant 2\varepsilon |z| e^{2l|z|}.$$

Lemma 6. Let

$$W(z) = \int_0^l e^{2iz\,dH(t)}, \qquad W_k(z) = \int_{t_{k-1}}^{t_k} e^{2iz\,dH(t)} \quad (0 = t_0 < t_1 < \cdots < t_n = l),$$

σ be the type of the matrix-function $W(z)$, and σ_k the type of the matrix-function $W_k(z)$. If for no $x_1, x_2 \in [0, l]$ is the matrix-function

$$\int_{x_1}^{x_2} e^{2iz\,dH(t)}$$

scalar, then

$$\sigma = \sum_{k=1}^{n} \sigma_k.$$

Theorem 2. If for no $x_1, x_2 \in [0, l]$ is the matrix-function

$$\int_{x_1}^{x_2} e^{2iz\,dH(t)}$$

scalar, then the matrix-function

$$W(z) = \int_0^l e^{2iz\,dH(t)}$$

is ordered.

Proof. Let ε be an arbitrary positive number ($\varepsilon < 1/2,\ l/2$). Using the well-known theorem of N. N. Luzin, construct a projection matrix-function $P_1(t)$, continuous on the segment $[0, l]$, for which

$$mE(P \neq P_1) \leq \varepsilon^2,$$

and let

$$W_1(z) = \int_0^l e^{2iz\,dH_1(t)}$$

$$\left(H_1(t) = \int_0^t P_1(x)\,dx\right).$$

Choose numbers $\delta > 0$ and $N > l/\delta$ so that the inequalities

$$\|P_1(t') - P_1(t'')\| < \varepsilon \quad (|t' - t''| < \delta,\ t', t'' \in [0, l]); \tag{3}$$

$$\varepsilon l/n < e^{-2\varepsilon l/n} - e^{-4\varepsilon l/n} \quad (n > N), \tag{4}$$

hold, and consider the partition of the segment $[0, l]$ into n equal parts $(n > N)$ by the points

$$t_k = kl/n \quad (k = 0, 1, \dots, n).$$

Introducing the notation

$$W_k(z) = \int_{t_{k-1}}^{t_k} e^{2iz\,dH(t)}, \qquad W_k^{(1)}(z) = \int_{t_{k-1}}^{t_k} e^{2iz\,dH_1(t)}, \qquad E_k = E \cap [t_{k-1}, t_k], \quad \varepsilon_k = mE_k,$$

and using Lemmas 4 and 5, we obtain the estimates

$$\|W_k^{(1)}(-i)\| \geq e^{(1-\varepsilon)2l/n}, \qquad \|W_k(-i) - W_k^{(1)}(-i)\| \leq 2\varepsilon_k e^{2l/n}. \tag{5}$$

Let $\varepsilon_{k_j} < \varepsilon l/2n$ for $j = 1, 2, \dots, s$, and $\varepsilon_{k_j} \geq \varepsilon l/2n$ for $j = s+1, \dots, n$. Since

$$\varepsilon^2 \geq \sum_{j=s+1}^{n} \varepsilon_{k_j} \geq \frac{\varepsilon l}{2n}(n - s),$$

it follows that

$$s \geq \left(1 - \frac{2\varepsilon}{l}\right) n.$$

Putting $j = 1, 2, \dots, s$ and using inequalities (4) and (5), we obtain

$$\|W_{k_j}(-i)\| \geq \|W_{k_j}^{(1)}(-i)\| - \|W_{k_j}^{(1)}(-i) - W_{k_j}(-i)\| \geq e^{(1-\varepsilon)2l/n} - 2\varepsilon_{k_j} e^{2l/n}$$

$$> e^{(1-2\varepsilon)2l/n}\left[e^{2\varepsilon l/n} - \frac{\varepsilon l}{n} e^{4\varepsilon l/n}\right] > e^{(1-2\varepsilon)2l/n},$$

whence it follows* that

$$\sigma_{k_j} \geq (1 - 2\varepsilon)\frac{2l}{n} \quad (j = 1, 2, \dots, s),$$

where σ_{k_j} is the type of the matrix-function $W_{k_j}(z)$. By Lemma 6,

$$\sigma = \sum_{j=1}^{n} \sigma_{k_j} \geq \sum_{j=1}^{s} \sigma_{k_j} \geq (1 - 2\varepsilon)\frac{2l}{n}\left(1 - \frac{2\varepsilon}{l}\right) n = 2l(1 - 2\varepsilon)\left(1 - \frac{2\varepsilon}{l}\right),$$

and since, by (2), $\sigma \leq 2l$, and $\varepsilon > 0$ is arbitrary, $\sigma = 2l$, and the orderedness of the matrix-function $W(z)$ follows from M. S. Brodskii' s criterion (2).

3. Consider the canonical representation (2) of the unordered matrix-function

$$W(z)=\int_0^l e^{2iz\,dH(t)}.$$

Using Theorem 2, one can construct (a finite or infinite) system of pairwise nonintersecting intervals $\Delta_k=(a_k,b_k)$, possessing the following properties:

$$1)\ \sum_k (b_k-a_k)=m_0; \qquad 2)\ \int_{a_k}^{b_k} e^{2izt}\,dH(t)=e^{i(b_k-a_k)z}I.$$

Let F be a set of positive measure belonging to the segment $[0,l]$, $\mu(t)=m(F\cap[0,t])$. Introduce the notation $P_F(t)=P(\nu(t))$, where $\nu(t)$ is the function inverse to $\mu(t)$.

Theorem 3. If F_0 is the set obtained from the segment $[0,l]$ by discarding all the intervals Δ_k, then $mF_0=l_0$ and $P_{F_0}(t)\equiv P_0(t)$.

We shall say that the matrix-function $P(t)$ has an orthogonally symmetric structure on the segment $[a,b]$ if there exist pairwise nonintersecting intervals $\Delta_{mk}=(a_{mk},b_{mk})\subset[a,b]$ $(k=1,2,\dots,n,\ n\geq 1;\ m=1,2)$, satisfying the conditions: 1) $b_{1k}-a_{1k}=b_{2k}-a_{2k}=d_k$

$$(k=1,2,\dots,n), \qquad \sum_{k=1}^{n} 2d_k=b-a;$$

2) between Δ_{1k} and Δ_{2k} there are no intervals Δ_{ms} $(s>k)$; 3) $P(a_{1k}+t)P(b_{2k}-t)\equiv 0$ $(0\leq t\leq d_k,\ k=1,2,\dots,n)$.

Theorem 4. In order that the matrix-function

$$W(z)=\int_a^b e^{2iz\,dH(t)} \qquad \left(H(t)=\int_0^t P(x)\,dx,\ 0\leq a<b\leq l\right)$$

be scalar, it is necessary and sufficient that there exist a sequence of sets $F_1\subseteq F_2\subseteq\dots\subseteq[a,b]$ such that $\lim_{k\to\infty} mF_k=b-a$, and each of the matrix-functions $P_{F_k}(t)$ has an orthogonally symmetric structure on the segment $[0,mF_k]$.

4. In the Hilbert space $\mathcal{L}^2(0,l)$ consider the integral operator

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* Since the matrix-function $W_{k_j}(z)$ is unitary on the real axis, it follows from (3) that for all $y<0$

$$\|W_{k_j}(x+iy)\|\leq e^{-\sigma_{k_j}y}.$$

$$Af = 2i \int_x^l f(t)\xi(t)\,dt\,\xi^*(x) \quad (\xi(x) = \|\varphi_1(x), \varphi_2(x)\|,\ \varphi_k \in \mathcal{L}^2(0,l),$$

$$\xi(x)\xi^*(x) \equiv 1).$$

The operator A belongs to the class Ω_2^+, and the matrix-function

$$W(z) = \int_0^l e^{2iz\,dH(t)} \qquad \left(H(t) = \int_0^t P(x)\,dx,\ P(x) = \xi^*(x)\xi(x)\right)$$

is characteristic for it. Let us note criteria for the unicellularity of the operator A that follow from the results obtained above.

Theorem 5. If the functions $\varphi_k(x)$ $(k = 1, 2)$ are continuous on the segment $[0, l]$, then the operator A is unicellular.

Theorem 6. If the functions $\varphi_k(x)$ $(k = 1, 2)$ are piecewise continuous on the segment $[0, l]$, then for the unicellularity of the operator A it is necessary and sufficient that, for each point of discontinuity t_{ki} of the function $\varphi_k(x)$, the measure of the set of all t from the interval $(0, \delta)$ for which $\xi(t_{ki} - t)\xi^*(t_{ki} + t) \neq 0$ be positive for every $\delta > 0$.

Theorem 7. If in some neighborhood of each point $t \in (0, l)$ the vector-function $\xi(t)$ has no mutually orthogonal values, then the operator A is unicellular.

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Note: Figure translations are in progress. See original paper for figures.

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