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1964

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Abstract

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A Difference Method for Solving Stefan-Type Problems for a Quasilinear Parabolic Equation with Discontinuous Coefficients

(Presented by Academician A. A. Dorodnitsyn on 10 February 1964)

In the present note we consider two-phase and one-phase Stefan-type problems for a quasilinear equation of parabolic type with discontinuous coefficients under nonlinear boundary conditions. These problems include spatial Stefan-type problems with radial symmetry in the cylindrical and spherical cases. In writing the present note, the method of the papers (1-6) has been used and generalized.

1°. Consider the following two-phase Stefan-type problem: find the functions $\bar{u}(x,t)$, $u(x,t)$, $y(t)$ satisfying the conditions:

$$(\bar{k}(x)\bar{u}_x)_x - \bar{\rho}(x,t,\bar{u})\bar{u}_t = 0, \qquad 0 < x < y(t), \quad x \neq l_m, \quad t > 0; \tag{1}$$

$$[\bar{u}(x,t)]_{x=l_m} = 0, \qquad [\bar{k}(x)\bar{u}_x(x,t)]_{x=l_m} = 0, \qquad t > 0; \tag{2}$$

$$\bar{k}(0)\bar{u}_x(0,t) = -\bar{q}(t,\bar{u}(0,t)), \qquad \bar{u}(y(t),t) = 0, \qquad t > 0; \tag{3}$$

$$\bar{u}(x,0) = \bar{\varphi}(x), \qquad 0 \leq x \leq c = y(0); \tag{4}$$

$$(k(x)u_x)_x - \rho(x,t,u)u_t = 0, \qquad y(t) < x < l, \quad x \neq l_m, \quad t > 0; \tag{5}$$

$$[u(x,t)]_{x=l_m} = 0, \qquad [k(x)u_x(x,t)]_{x=l_m} = 0, \qquad t > 0; \tag{6}$$

$$k(l)u_x(l,t) = -q(t,u(l,t)), \qquad u(y(t),t) = 0, \qquad t > 0; \tag{7}$$

$$u(x,0) = \varphi(x), \qquad c \leq x \leq l; \tag{8}$$

$$\gamma(y(t),t)y'(t) = k(y(t))u_x(y(t),t) - \bar{k}(y(t))\bar{u}_x(y(t),t) + \Phi(y(t),t) \tag{9}$$

for $t > 0$; $y(0) = c$,

where $\bar{k}(x)$ and $k(x) \in Q$ $(x \neq l_m)$ for $0 \leq x \leq l$; $\bar{\rho}(x,t,s)$ and $\rho(x,t,s) \in Q$ $(x \neq l_m)$ in $G = \{0 \leq x \leq l,\ t \geq 0,\ |s| < \infty\}$*; $[z(x,t)]_{x=l_m} =$

$$= z(l_m + 0, t) - z(l_m - 0, t), \quad m = 1, 2, \dots, m_1 - 1; \quad 0 = l_0 \leq l_1 \leq \cdots$$

$$\leq \cdots l_{m_0 - 1} \leq c < l_{m_0} \leq l_{m_0 + 1} \leq \cdots \leq l_{m_1} = l, \quad m_1 \geq m_0 \geq 1$$

(the case $y(0) = 0$ is possible); $\bar{q}(t,\bar{u})$, $q(t,u)$, $\gamma(x,t)$, $\Phi(x,t)$, $\bar{\varphi}(x)$, $\varphi(x)$ are given functions; $\bar{\varphi}(c) = \varphi(c) = 0$. We note that the two-phase Stefan-type problem for more general equations of the form

$$(\bar{\nu}(r)\bar{\lambda}(\bar{v})\bar{v}_r)_r - \bar{c}(r,t,\bar{v})\bar{v}_t = 0, \qquad r_0 < r < R(t), \quad r \neq r_m, \quad t > 0;$$

$$(\nu(r)\lambda(v)v_r)_r - c(r,t,v)v_t = 0, \qquad R(t) < r < r_m, \quad r \neq r_m, \quad t > 0,$$

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* We shall write $f(x, z_1, \dots, z_n) \in C^{(s_0, s_1, \dots, s_n)}$ in some domain G of the variables $(x, z_1, \dots, z_n)$ if all derivatives of the form

$$\partial^{p_0+p_1+\cdots+p_n} f / \partial x^{p_0} \partial z_1^{p_1} \cdots \partial z_n^{p_n}, \quad 0 \leq p_i \leq s_i, \quad 0 \leq p_0 + p_1 + \cdots + p_n \leq \max\{s_0, s_1, \dots, s_n\}$$

exist and are continuous jointly in their arguments everywhere in G. If, however, the indicated derivatives are continuous jointly in their arguments everywhere in G except, possibly, at points of the form $(x = l_m, z_1, \dots, z_n) \in G$ $(m = 1, 2, \dots, m_1 - 1)$, where these derivatives may have jump discontinuities, then we shall write

$$f(x, z_1, z_2, \dots, z_n) \in Q^{(s_0, s_1, \dots, s_n)} \quad (x \neq l_m).$$

In the case $s_0 = s_1 = \cdots = s_n = 0$ we denote

$$C^{(0,0,\dots,0)} = C, \qquad Q^{(0,0,\dots,0)} = Q.$$

where $\bar{v}(r), v(r), \bar{c}(r,t,\bar{v}), c(r,t,v) \in Q(r \neq r_m), r_0 > 0$, is reduced to problem (1)–(9) by means of the substitution

$$x = r - r_0, \quad y(t) = R(t) - r_0, \quad \bar{u}(x,t) = \int_0^{\bar{v}(r,t)} \bar{\lambda}(s)\, ds, \quad u(x,t) = \int_0^{v(r,t)} \lambda(s)\, ds, \quad \bar{k}(x) = \bar{v}(r), \quad k(x) = v(r)$$

Definition. We shall call the functions $\bar{u}(x,t)$, $u(x,t)$, $y(t)$ a solution of problem (1)–(9) if: 1) $y(t)$ is defined and continuous on some interval $0 \leq t \leq T$ and has a continuous derivative $y'(t)$ for $0 < t \leq T$; 2) $\bar{u}(x,t)$ is defined and continuous in $\{0 \leq x \leq y(t),\ 0 \leq t \leq T\}$; the derivative $\bar{u}_x(x,t) \in Q(x \neq l_m)$ and is uniformly bounded in $\{0 \leq x \leq y(t),\ 0 < t \leq T\}$; the derivatives $\bar{u}_{xx}(x,t)$, $\bar{u}_t(x,t)$ are continuous and uniformly bounded in $\{0 < x < y(t),\ x \neq l_m,\ 0 <$

$t \leq T\}$ $(m = 1, 2, \dots, m_1 - 1)$; the function $u(x,t)$ has analogous properties in $\{y(t) \leq x \leq l,\ 0 \leq t \leq T\}$; 3) all conditions (1)–(9) are satisfied.

For an approximate solution of problem (1)–(9), assuming $y(t)$ monotone, we introduce a rectangular difference grid: we divide the interval $0 \leq x \leq l$ by points x_i into N equal parts with step h, where $x_i = ih,\ i = 0, 1, \dots, N;\ x_N = l;\ x_{i_m} = i_{mh} = l_m\ (m = 1, 2, \dots, m_1 - 1);\ c = i_{ch}$, and the time step τ_n will be chosen so that for each

$$t = t_n = \sum_{k=1}^{n} \tau_k \quad (t_0 = 0)$$

the end of a segment of the polygonal line approximating the curve $y(t)$ falls on the node with coordinates $(y_n, t_n),\ y_n = c + nh,\ n = 0, 1, \dots$. We replace problem (1)–(9) by the following difference problem for determining τ_n and the approximate values $\bar{w}_{in}, w_{in}$ of the functions $\bar{u}(x,t), u(x,t)$ at the nodes (x_i, t_n):

$$\delta_{\bar{x}} \left(\bar{k}_i \delta_x \bar{w}_{in}\right) - \bar{\rho}_{i,n-1} \delta_t^- \bar{w}_{in} = 0, \quad 1 \leq i \leq i_c + n - 1, \quad i \neq i_m \tag{10}$$

$$(m = 1, 2, \dots, m_0 - 1), \quad n = 1, 2, \dots;$$

$$\delta_{\bar{x}} \left(\bar{k}_{i_m} \delta_x \bar{w}_{i_m,n}\right) = 0 \quad (m = 1, 2, \dots, m_0 - 1), \quad n = 1, 2, \dots; \tag{11}$$

$$\bar{k}_0 \delta_x \bar{w}_{0,n} = -\bar{q}_{n-1}, \quad \bar{w}_{i_c+n,n} = 0, \quad n = 1, 2, \dots; \tag{12}$$

$$\bar{w}_{i0} = \bar{\varphi}_i = \bar{\varphi}(x_i), \quad 0 \leq i \leq i_c; \tag{13}$$

$$\delta_{\bar{x}} \left(k_i \delta_x w_{in}\right) - \rho_{i,n-1} \delta_t^- w_{in} = 0, \quad i_c + n + 1 \leq i \leq N - 1; \tag{14}$$

$$i \neq i_m\ (m = m_0, m_0 + 1, \dots, m_1 - 1), \quad n = 1, 2, \dots;$$

$$\delta_{\bar{x}} \left(k_{i_m} \delta_x w_{i_m,n}\right) = 0 \quad (m = m_0, \dots, m_1 - 1), \quad n = 1, 2, \dots; \tag{15}$$

$$k_{N-1} \delta_x w_{N-1,n} = -q_{n-1}, \quad w_{i_c+n,n} = 0, \quad n = 1, 2, \dots; \tag{16}$$

$$w_{i0} = \varphi_i = \varphi(x_i), \quad i_c \leq i \leq N; \tag{17}$$

$$\gamma_{n-1} \frac{h}{\tau_n} = k(y_n) \delta_x w_{i_c+n,n} - \bar{k}(y_{n-1}) \delta_x \bar{w}_{i_c+n-1,n} + \Phi_{n-1}, \quad n = 1, 2, \dots, \tag{18}$$

where the following notation has been adopted:

$$\bar{k}_i = \bar{k}(x_i), \quad k_i = k(x_i), \quad \bar{\rho}_{i,n-1} = \bar{\rho}(x_i, t_{n-1}, \bar{w}_{i,n-1}),$$

$$\rho_{i,n-1} = \rho(x_i, t_{n-1}, w_{i,n-1}), \quad \bar{q}_{n-1} = \bar{q}(t_{n-1}, w_{0,n-1}),$$

$$q_{n-1} = q(t_{n-1}, w_{N,n-1}), \quad \gamma_{n-1} = \gamma(y_{n-1}, t_{n-1}), \quad \Phi_{n-1} = \Phi(y_{n-1}, t_{n-1}),$$

$$\delta_x z_{in} = \frac{1}{h}(z_{i+1,n} - z_{in}), \quad \delta_{\bar{x}} z_{in} = \frac{1}{h}(z_{in} - z_{i-1,n}), \quad \delta_t^- z_{in} = \frac{1}{\tau_n}(z_{in} - z_{i,n-1}).$$

System (10)–(18) is nonlinear with respect to the unknowns $\bar{w}_{in}, w_{in}, \tau_n$, and the following iteration method is proposed for its solution. Let $\bar{w}_{ik}, w_{ik}, \tau_k$

$(k = 0, 1, \ldots, n-1)$ (τ_0 is not determined), satisfying (10)–(18), be known. Then the successive approximations $\bar{w}_{in}^{(s)}, w_{in}^{(s)}, \tau_n^{(s)}$,

For $s = 0, 1, 2, \ldots$, we shall determine the quantities $\bar{w}_{in}, w_{in}, \tau_n$ $(n \geq 1)$ in the following way. Set $\tau_n^{(0)} > 0$, and from (10)–(13) and (14)–(17), with $\tau = \tau_n^{(0)}$, find $\bar{w}_{in}^{(0)}, w_{in}^{(0)}$, while from (19)

$$\tau_n^{(s+1)} = \frac{1}{\Phi_{n-1} + \bar{q}_{n-1} - q_{n-1}} \left[h\gamma_{n-1} + \tau_n^{(s)} \left(\bar{q}_{n-1} + \bar{k}(y_{n-1}) \delta_x \bar{w}_{i_c+n-1,n}^{(s)} - q_{n-1} - k(y_n) \delta_x w_{i_c+n,n}^{(s)} \right) \right] \tag{19}$$

for $s = 0$ we obtain $\tau_n^{(1)}$. In general, knowing $\tau_n^{(s)} > 0$, from (10)–(13) and (14)–(17), with $\tau = \tau_n^{(s)}$, we find $\bar{w}_{in}^{(s)}, w_{in}^{(s)}$, and then from (19) obtain $\tau_n^{(s+1)}$, etc. In solving the systems (10)–(13) and (14)–(17) it is convenient to use the sweeping method (7).

Theorem 1. Let the following conditions be satisfied: 1) $\bar{\varphi}(x) \in C$ on $[0, c]$, $\bar{\varphi}(c) = 0$, $-\bar{q}(0, \bar{\varphi}(0)) \leq \bar{k}_i \delta_x \bar{\varphi}_i \leq 0$, $\delta_{\bar{x}}^{-}(\bar{k}_i \delta_x \bar{\varphi}_i) \geq 0$; $\varphi(x) \in C$ on $[c, l]$, $\varphi(c) = 0$, $-\bar{q}(0, \bar{\varphi}(0)) - \Phi(c, 0) \leq k_i \delta_x \varphi_i \leq -q(0, \varphi(l))$, $\delta_{\bar{x}}^{-}(k_i \delta_x \varphi_i) \geq 0$; 2) $\bar{q}(t, \bar{u}) \geq 0$, $\bar{q}_t \geq 0$, $\bar{q}_{\bar{u}} \geq 0$ for $t \geq 0$, $0 \leq \bar{u} \leq l_{m_0} \bar{q}_{\max} / \bar{k}_{\min}$; $q(t, u) \geq 0$, $q_t \leq 0$, $q_u \leq 0$ for $t \geq 0$, $\varphi(l) \leq u \leq 0$; 3) $\bar{k}(x) \geq \bar{k}_{\min} > 0$, $\bar{k}(x) \in Q$ $(x \neq l_m)$, $\bar{\rho}(x, t, \bar{u}) \geq \bar{\rho}_{\min} > 0$, $\bar{\rho}(x, t, \bar{u}) \in Q$ $(x \neq l_m)$ $(m = 1, 2, \ldots, m_0 - 1)$ in $G_2 = \{0 \leq x \leq l_{m_0},\ t \geq 0,\ 0 \leq \bar{u} \leq l_{m_0} \bar{q}_{\max} / \bar{k}_{\min}\}$; $k(x) \geq k_{\min} > 0$, $k(x) \in Q$ $(x \neq l_m)$, $\rho(x, t, u) \geq \rho_{\min} > 0$, $\rho(x, t, u) \in Q$ $(x \neq l_m)$ $(m = m_0, \ldots, m_1 - 1)$ in $G_1 = \{c \leq x \leq l,\ t \geq 0,\ \varphi(l) \leq u \leq 0\}$.

Then, for any choice of $\tau_n^{(0)} > 0$, the iterations (10)–(17), (19) are uniquely determined for all $s = 0, 1, 2, \ldots$, and, as $s \to \infty$, the quantities $\bar{w}_{in}^{(s)}, w_{in}^{(s)}, \tau_n^{(s)}$, changing monotonically, will converge to the solution of the system (10)–(18).

The proof of this theorem is carried out approximately according to the same scheme as the proof of analogous theorems in ([2,4,5]).

Along with the conditions of Theorem 1, consider the following conditions:

a) $\bar{q}(t, \bar{u}) \equiv \text{const} = \bar{q}$, $q(t, u) \equiv \text{const} = q$; $\bar{k}_0 \delta_x \bar{\varphi}_0 = -\bar{q}$, $k_{N-1} \delta_x \varphi_{N-1} = -q$, $\delta_{\bar{x}}^{-}(\bar{k}_i \delta_x \bar{\varphi}_i) \leq \mu \bar{\rho}(x_i, 0, \bar{\varphi}_i)$, $\delta_{\bar{x}}^{-}(\bar{k}_{i_m} \delta_x \bar{\varphi}_{i_m}) = 0$ $(m = 1, 2, \ldots, m_0 - 1)$, $\delta_{\bar{x}}^{-}(k_i \delta_x \varphi_i) \leq \mu \rho(x_i, 0, \varphi_i)$, $\delta_{\bar{x}}^{-}(k_{i_m} \delta_x \varphi_{i_m}) = 0$ $(m = m_0, \ldots, m_1 - 1)$ for all sufficiently small $h > 0$; $\bar{\mu}, \mu$ are constants;

b) $\bar{k}(x) \in Q^{(5)}$ $(x \neq l_m)$, $\bar{\varphi}(x) \in Q^{(2)}$ $(x \neq l_m)$, $\bar{\rho}(x, t, u) \in Q^{(4,0,4)} \cap Q^{(0,1,0)}$ $(x \neq l_m)$, $\bar{\rho}_t \geq 0$, $\bar{\rho}_{\bar{u}} \geq 0$ in G_2; $k(x) \in Q^{(5)}$ $(x \neq l_m)$, $\varphi(x) \in Q^{(2)}$ $(x \neq l_m)$, $\rho(x, t, u) \in Q^{(4,0,4)} \cap Q^{(0,1,0)}$ $(x \neq l_m)$, $\rho_t \geq 0$, $\rho_u \geq 0$ in G_1 $(m = 1, 2, \ldots, m_1 - 1)$;

c) $\min(\Phi + \bar{q} - q) > l_{m_0 - 1} M_2^0 \bar{\rho}_{\max} + (l_{m_0} - l_{m_0 - 1}) \max\{\bar{M}_2^0 \bar{\rho}_{\max};\ M_2^0 \rho_{\max}\} + (l - l_{m_0}) M_2 \rho_{\max}$, where $\bar{M}_2 = \max\{\bar{\mu}; \Lambda_1 \bar{q} / \bar{k}_{\min}\}$, $M_2 = \max\{\mu; \Lambda_1 (\bar{q} + \Phi_{\max}) / k_{\min}\}$, $\Lambda_1 = \max[(\Phi + \bar{q} - q)/\gamma]$;

d) $\gamma(x,t) \in C^{(0,1)}$, $\Phi(x,t) \in C$ in $\Delta = \{c \leq x \leq l_{m_0},\ t \geq 0\}$.

If the conditions of Theorem 1 and conditions a)–d) are fulfilled, there is a sequence h_ν, $\nu = 1, 2, ...$, such that the polygonal lines $y(t, h_\nu)$, obtained by joining by straight-line segments the points (y_n, t_n), as $\nu \to \infty$, converge uniformly on some interval $0 \leq t \leq T$ to a monotone curve $y(t)$, $c \leq y(t) \leq l_{m_0}$, and there also exist solutions $\bar{u}(x,t)$, $u(x,t)$ of the boundary-value problems (1)–(4) and (5)–(8) for this curve $y(t)$. Suppose the inequalities

e) $|u_t(x',t) - u_t(x'',t)| \leq M(\delta,\gamma)|x' - x''|^\alpha$ in $\{y(t) + \delta \leq x', x'' \leq l - \delta,\ \gamma \leq t \leq T\}$ and $|u_t(x',t) - u_t(x'',t)| \leq M(\delta,\gamma)|x' - x''|^\alpha$ in $\{\delta \leq x', x'' \leq y(t) - \delta,\ \gamma \leq t \leq T\}$, where $l_m \leq x', x'' \leq l_{m+1}$ $(m = 0, 1, \ldots, m_1 - 1)$, for all sufficiently small $\delta > 0$, $\gamma > 0$; $0 < \alpha = \alpha(\delta,\gamma) \leq 1$.

Theorem 2. If the hypotheses of Theorem 1 and conditions a)–e) are satisfied, then there exists a solution of problem (1)–(9), which can be obtained as the limit, as $h_\nu \to 0$, of solutions of the difference problem (10)–(18).

Using Theorem 2 and the method and results of papers ([8–10]), one can prove the following two theorems:

Theorem 3. If $\bar{\rho} = \bar{\rho}(x,t)$, $\rho = \rho(x,t)$ and all the hypotheses of Theorem 1 and conditions a)–d) are satisfied, then there exists a solution of problem (1)–(9).

Theorem 4. If all the hypotheses of Theorem 1 and conditions a)–d) are satisfied with the classes Q replaced by the classes C (i.e. $m_1 = m_0 = 1$), then there exists a solution of the Stefan-type problem (1)–(9).*

2°. Consider the one-phase Stefan-type problem: to find the functions $u(x,t)$, $y(t)$ from conditions (5)–(8) and condition ([1,3,5]) (cf. (11)):

$$\gamma(y(t),t)y'(t) = k(y(t))u_x(y(t),t) + \Phi(y(t),t), \qquad y(0) = c. \tag{20}$$

For an approximate solution of problem (5)–(8), (20), one may use the difference scheme (14)–(18) and the iteration method, putting in (18) and (19) $\bar{w}_{in} \equiv \bar{w}_{in}^{(s)} \equiv \bar{q} \equiv 0$. Theorems analogous to Theorems 1–4 hold; their statements may be obtained from the corresponding Theorems 1–4 by putting in them $\bar{q} \equiv \bar{\varphi} \equiv \bar{u} \equiv \bar{w}_{in} \equiv \bar{w}_{in}^{(s)} \equiv \bar{M}_2 \equiv 0$. We note that, in problem (5)–(8), (20), the condition $u(y(t),t) \equiv 0$ may be replaced by the more general one: $u(y(t),t) = \psi(y(t),t)$, where $\psi(x,t)$ is a prescribed function.

3°. In an analogous way, another one-phase Stefan-type problem is considered: to find the functions $\bar{u}(x,t)$, $y(t)$ from conditions (1)–(4) and ([5,6]) (cf. ([12,13])):

$$\gamma(y(t),t)y'(t) = -\bar{k}(y(t))\bar{u}_x(y(t),t) + \Phi(y(t),t), \qquad y(0) = c. \tag{21}$$

4°. In contrast to papers ([1–5,12,13]), the uniqueness of the solutions of all the Stefan-type problems considered above in the class of sufficiently smooth solutions, for sufficiently smooth coefficients of the equations, can be proved without the assumption of monotonicity of $y(t)$. In particular, the following is true.

Theorem 5. If $\bar{q} = \bar{q}(t)$, $q = q(t)$, $\bar{\rho} = \bar{\rho}(x,t)$, $\rho = \rho(x,t)$, $\Phi(x,t) \in C^{(1,0)}$ in Δ, and conditions b) and d) are satisfied, then problem (1)–(9) has no more than one solution.

The author expresses deep gratitude to B. M. Budak for proposing the problem and for attentive guidance in the execution of the present work, and also to A. B. Uspenskii for useful discussion of the results.

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Received
8 II 1964

REFERENCES

1. F. P. Vasil' ev, *Zhurn. vychislit. matem. i matem. fiz.*, **3**, No. 5 (1963).

2. F. P. Vasil' ev, A. B. Uspenskii, *Zhurn. vychislit. matem. i matem. fiz.*, **3**, No. 5 (1963).

3. F. P. Vasil' ev, *DAN*, **152**, No. 4 (1963).

4. F. P. Vasil' ev, A. B. Uspenskii, *DAN*, **152**, No. 5 (1964).

5. B. M. Budak, F. P. Vasil' ev, A. B. Uspenskii, Collection, Computing Center of Moscow Univ., *Numerical Methods in Gas Dynamics*, issue 3, 1964.

6. J. Douglas, T. M. Gallie, *Duke Math. J.*, **22**, No. 4 (1955).

7. I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, vol. 2, Moscow, 1962.

8. O. A. Oleinik, *Izv. AN SSSR, ser. matem.*, **25**, No. 1 (1961).

9. A. M. Il' in, A. S. Kalashnikov, O. A. Oleinik, *UMN*, **17**, issue 3 (1962).

10. A. A. Samarskii, *DAN*, **121**, No. 2 (1958).

11. L. I. Rubinshtein, *DAN*, **142**, No. 3 (1962).

12. J. Douglas, *Proc. Am. Math. Soc.*, **8**, No. 2 (1957).

13. W. T. Kyner, *J. Math. and Mech.*, **8**, No. 4 (1959).

* We note that analogous theorems from ([4,5]), which are a special case of Theorem 4, were proved under the additional assumption of boundedness of the fourth derivatives of the solutions $\bar{u}(x,t)$, $u(x,t)$ with respect to the variable x.

Note: Figure translations are in progress. See original paper for figures.