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Abstract

Full Text

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A UNIVERSAL BICOMPACTUM OF GIVEN WEIGHT AND GIVEN DIMENSION

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A. The main purpose of the work is the following

Main Theorem 1. *For every infinite cardinal τ and every natural number n there exists a bicompactum $B_{n\tau}$ of weight τ and dimension $\dim B_{n\tau} = n$, containing a topological image of every completely regular space X of weight $\leq \tau$ and dimension $\dim X \leq n$.*

This theorem is a positive answer to one problem of P. S. Aleksandrov* (([1]), pp. 50–51). From it follows the well-known theorem of E. G. Sklyarenko ([7]) on the existence of a bicompact extension bX of the same weight and the same dimension as the original space X. For countable weight, universal n-dimensional compacta were first constructed (without proof) by K. Menger in 1926. The existence of universal n-dimensional compacta was proved by Nöbeling in 1930 (([3]), pp. 89 and 94). In the case of uncountable weight some partial results were obtained by B. Pasynkov ([6]) by the method of spectra.

Remark. The dimension $\dim X$ of a normal space X is the dimension defined in the usual way by means of finite open coverings; as Yu. M. Smirnov indicated (([8]), pp. 297–298), the dimension $\dim X$ of a completely regular space X is expediently defined in the same way by means of finite normal coverings. For normal spaces such a definition of the dimension dim is equivalent to the usual one.

Main Theorem 1′. *For every infinite cardinal τ and every natural number n there exists a normed ring $R_{n\tau}$ of weight τ and algebraic dimensional rank* $\operatorname{ard} R_{n\tau} = n$***, which can be continuously-homomorphically mapped onto some regular subring of the ring*** $C(X)$ ***of all functions for every such space*** X of weight $\leq \tau$, for which $\operatorname{ard} C(X) \leq n$.*

Theorem 1_{B}. *The space $B(R)$ of h-ideals** of the ring $R = R_{n\tau}$ is the required universal bicompactum $B_{n\tau}$.*

B. Construction of the ring $R_{n\tau}$ and some known facts

Nöbeling proved that the set M_n^{2n+1} of all such points of the unit $(2n+1)$-dimensional cube I^{2n+1} which have no more than n rational coordinates is a universal space for all spaces of countable weight and dimension $\leq n$. Moreover, the set of all topological mappings of each such space X into the set M_n^{2n+1} is dense in the space $C(X, I^{2n+1})$ of all continuous mappings of the space X into the cube I^{2n+1} (([3]), pp. 94–95).

Since $\dim M_n^{2n+1} = n$, by a theorem of M. Katětov ([4]) there exists a uniformly zero-dimensional**** mapping of the set M_n^{2n+1} into the unit the n-dimensional cube I^n. Fix, for all that follows, the number n and n functions $\varphi_1, \dots, \varphi_n$—superpositions of the mapping φ and projections of the cube I^n onto its edges, and also $n+1$ functions $\pi_1, \dots, \pi_{n+1}$—projections of the set M_n^{2n+1} onto the first $n+1$ edges of the cube I^{2n+1}. Since the functions $\pi_i(z)$ are uniformly continuous and $\dim M_n^{2n+1} = n$, it follows from a theorem of J. Nagata ([5]) that, for every natural number k and every $i = 1, 2, \dots, n+1$, there exist polynomials $\hat{P}_{ik}(t_1, \dots, t_{m(i,k)})$ and $\hat{Q}_{ikl}(t_1, \dots, t_n)$ with real coefficients, where $l = 1, 2, \dots, m(i,k)$, and functions $\rho_{ikl}(z)$, defined on M_n^{2n+1}, such that the following conditions are fulfilled:

$$\left|\pi_i(z) - \hat{P}_{ik}(\rho_{ik1}(z), \dots, \rho_{ikm(i,k)}(z))\right| < \frac{1}{k}$$

and

$$\rho_{ikl}^2(z) = \rho_{ikl}(z) \cdot \hat{Q}_{ikl}(\varphi_1(z), \dots, \varphi_n(z))$$

for all z and k.

Construction. Fix additionally, for all that follows, the series of polynomials $\hat{P}_{ik}$ and $\hat{Q}_{ikl}$, and, moreover, an infinite cardinal τ and a set of indices $\{\alpha\}$ of cardinality τ. We construct, by induction, an increasing countable sequence of sets Π_p. Let $\Pi_0 = \{\mathfrak{D}_\alpha\}$ be a set, denoted by all indices α, of cardinality τ. The totality of all ordered systems λ_1 consisting of $n+1$ polynomials

$$R_1 = R_1(\mathfrak{D}_{\alpha_{1,1}}, \dots, \mathfrak{D}_{\alpha_{1,m(1)}}), \dots, R_{n+1} = R_{n+1}(\mathfrak{D}_{\alpha_{n+1,1}}, \dots, \mathfrak{D}_{\alpha_{n+1,m(n+1)}})$$

with rational coefficients in the variables $\mathfrak{D}_\alpha$ has cardinality τ; therefore, adjoining to the set Π_0 sets of elements $\xi_{ik}^{\lambda_1}$, $\rho_{ikl}^{\lambda_1}$, and $\varphi_{jk}^{\lambda_1}$ (denoted by all indices λ_1 and natural numbers $k = 1, 2, \dots$, $j = 1, \dots, n$, $i = 1, \dots, n+1$, $l = 1, \dots, m(i,k)$), we obtain a set

$$\Pi_1 = \{\mathfrak{D}_\alpha; \xi_{ik}^{\lambda_1}; \rho_{ikl}^{\lambda_1}; \varphi_{jk}^{\lambda_1}\}$$

* Very recently, an extremely simple solution of this problem was proposed by B. Pasynkov ([11]).

** See definitions 1, 3 and 4.

*** By a function we shall henceforth mean bounded continuous single-valued functions; by a space—a completely regular space.

**** A mapping f is uniformly zero-dimensional if for every $\varepsilon > 0$ there exists such $\delta > 0$ that, as soon as $\operatorname{diam} A < \delta$, the set $f^{-1}A$ decomposes into a sum of pairwise disjoint sets open in A of diameter less than ε.

of cardinality λ. Continuing this construction for each natural p, we obtain a set

$$\Pi_p = \{\mathfrak{D}_\alpha; \xi_{ik}^{\lambda_1}; \rho_{ikl}^{\lambda_1}; \varphi_{jk}^{\lambda_1}; \dots; \xi_{ik}^{\lambda_p}; \rho_{ikl}^{\lambda_p}; \varphi_{jk}^{\lambda_p}\},$$

where the index λ_p runs through the set of ordered systems of $n+1$ polynomials with rational coefficients in unknowns belonging to Π_{p-1}.

Finally, consider the free normed ring C with generators taken from the countable sum

$$\Pi = \bigcup_p \Pi_p,$$

consisting, by definition, of all polynomials with real coefficients in the variables from the set Π, in which the norm is defined by the following conventions:

a) $\|\mathfrak{D}_\alpha\| = \|\varphi_{jk}^{\lambda_p}\| = 1$; $\|\rho_{ikl}^{\lambda_p}\| = \|\hat{Q}_{ikl}\|$, where $\|\hat{Q}_{ikl}\|$ is the sum of the moduli of all coefficients of the polynomial $\hat{Q}_{ikl}$;

b) $\|\xi_{ik}^{\lambda_p}\| = \dfrac{2M(\lambda_p)}{k}$, where

$$M(\lambda_p) = \max\{1, \|R_1\|, \dots, \|R_{n+1}\|\},$$

if $\lambda_p = \{R_1, \dots, R_{n+1}\}$, and the norm of each element

$$R = \sum_{q=0}^{s} \sum_{q_1+\dots+q_r=q} a_{q_1\dots q_r} t_1^{q_1} \dots t_r^{q_r}$$

from C is given by the recurrent equality

$$\|R\| = \sum\sum |a_{q_1\dots q_r}|\, \|t_1\|^{q_1} \dots \|t_r\|^{q_r},$$

where $t_1, \dots, t_r \in \Pi$.

Theorem 1_R. *The factor ring $R = C/J$ of the ring C by the closed ideal J generated by all its elements of the form*

$$1) \quad \xi_{ik}^{\lambda_p} - R_i(t_{i1}, \dots, t_{im(i)}) + \hat{P}_{ik}(\rho_{ik1}^{\lambda_p}, \dots, \rho_{ikm(i,k)}^{\lambda_p})$$

and all its elements of the form

$$2) \quad (\rho_{ikl}^{\lambda_p})^2 - \rho_{ikl}^{\lambda_p} \cdot \hat{Q}_{ikl}(\varphi_{1k}^{\lambda_p}, \dots, \varphi_{nk}^{\lambda_p}),$$

where $\lambda_p = \{R_1, \dots, R_{n+1}\}$, $t_{i1}, \dots, t_{im(i)} \in \Pi_{p-1}$, is the required ring $R_{n\tau}$ from Theorem 1_B.

B. Proof of Theorem 1. It is not difficult to see that the weight of the ring $R = R_{n\tau}$ is τ. To prove the inequality $\operatorname{ard} R_{n\tau} \leq n$, we introduce the necessary definition.

Definition 1. The **algebraic-dimensional rank** $\operatorname{ard} G$ of a ring G is the least of all such natureal numbers n, such that for every system of $n+1$ elements $f_1, \dots, f_{n+1}$ of the ring G and every positive number ε there exist elements $g_1, \dots, g_{n+1}$ and a system Φ of n elements of the ring G such that $\|f_i - g_i\| < \varepsilon$ and $g_i \in D \operatorname{alg} D\Phi$ for each $i = 1, 2, \dots, n+1$. Here $D\Phi$ denotes the smallest of the subrings of the ring G containing Φ, and $\operatorname{alg} \Phi$ denotes the set of all such elements g of the ring G, each of which satisfies the equation $g^2 = g\varphi$ for some φ in Φ.

Proof. Indeed, let the elements $\tilde{f}_1, \dots, \tilde{f}_{n+1}$ belong to the factor ring $R = R_\pi$ of the ring C, let $f_i \in \tilde{f}_i$, where $1 \le i \le n+1$, and let $\varepsilon > 0$. It is not hard to find a natural number p and a system $\lambda_p = \{R_1, \dots, R_{n+1}\}$ such that $\|f_i - R_i\| < \varepsilon/2$ for each i. For the system λ_p we find a sufficiently large number k such that $4M(\lambda_p) < \varepsilon k$. Then, by a), b), and 1), for the corresponding elements $\xi_{ik}^{\lambda_p}$, $\rho_{ikl}^{\lambda_p}$, and $\varphi_{jk}^{\lambda_p}$, for arbitrary i, l, and j, we have*: $\|\tilde{R}_i - \tilde{g}_i\| = \|\tilde{\xi}_{ik}^{\lambda_p}\| \le \|\xi_{ik}^{\lambda_p}\| < \varepsilon/2$, where $\tilde{g}_i = \hat{P}_{ik}(\tilde{\rho}_{ik1}^{\lambda_p}, \dots, \tilde{\rho}_{ikm(i,k)}^{\lambda_p})$. By 2) we have: $\tilde{g}_i \in D \operatorname{alg} D\tilde{\Phi}$, where $\tilde{\Phi} = \{\tilde{\varphi}_{1k}^{\lambda_p}, \dots, \tilde{\varphi}_{nk}^{\lambda_p}\}$, and $\|\tilde{f}_i - \tilde{g}_i\| < \varepsilon$, as was required to prove.

Definition 2. The **dimensional rank** $\operatorname{rd} C(X)$ **of the ring of all functions on the space** X will be called the least of all such natural numbers n that for every system of $n+1$ functions $f_1 x, \dots, f_{n+1} x$ and every positive number ε there exist such functions $g_1 x, \dots, g_{n+1} x$ and a positive number δ that $|f_{ix} - g_{ix}| < \varepsilon$ for all x and

$$\bigcap_{i=1}^{n+1} g_i^{-1}[-\delta, \delta] = \emptyset.$$

Theorem 2. For any space X one always has

$$\operatorname{ard} C(X) = \operatorname{rd} C(X) = \dim X.$$

Theorem 2′. For any space X, $\dim X \le n$ if and only if $\operatorname{ard} G \le n$ for some subring G dense in $C(X)$ of the ring $C(X)$.

Lemma 1. If $\operatorname{ard} C(X) \le n$, then for any natural number k, any number $M \ge 1$, and every such system of $n+1$ functions $q_1 x, \dots, q_{n+1} x$, that $|q_{ix}| \le M$ for all i and x, there exist functions $\varphi_{jk} x$ and functions $\rho_{ikl} x$, where $j = 1, \dots, n$, $i = 1, \dots, n+1$, $l = 1, \dots, m(i,k)$, satisfying the following conditions:

$$\mathrm{a}_X) \quad \left| q_{ix} - M\hat{P}_{ik}(\rho_{ik1} x, \dots, \rho_{ikm(i,k)} x) \right| < 2M/k,$$

$$\mathrm{b}_X) \quad \rho_{ikl}^2 x = \rho_{ikl} x \cdot \hat{Q}_{ikl}(\varphi_{1k} x, \dots, \varphi_{nk} x)$$

and

$$\mathrm{c}_X) \quad |\varphi_{jk} x| \le 1$$

for all possible x, j, i, k, and l, where the polynomials $\hat{P}_{ik}$ and $\hat{Q}_{ikl}$ are defined above.

Proof of the main part of Theorem 1_R. Under the assumptions of Theorem 1_R there exists such a regular family $\Pi_0(X)$ of functions $\mathfrak{d}_\alpha x$ of cardinality τ (repetitions are not excluded!), that

$$\sup_x |\mathfrak{d}_\alpha x| \leq 1.$$

For each ordered system $\lambda_1 = \{q_1(x), \ldots, q_{n+1}(x)\}$, consisting of polynomials with rational coefficients in certain functions of the family $\Pi_0(X)$, we take the number

$$M(X, \lambda_1) = \max_i \{1, \|q_i\|\}$$

and find functions

$$\rho^{\lambda_1}_{ik1} x, \ldots, \rho^{\lambda_1}_{ikm} x \quad (m = m(i,k))$$

and functions

$$\varphi^{\lambda_1}_{1k} x, \ldots, \varphi^{\lambda_1}_{nk} x$$

in the ring $C(X)$, satisfying the conditions of Lemma 1 for $M = M(X, \lambda_1)$. Let

$$\xi^{\lambda_1}_{ik} x = q_{ix} - M \cdot \hat{P}_{ik}(\rho^{\lambda_1}_{ik1} x, \ldots, \rho^{\lambda_1}_{ikm} x).$$

Then

$$\sup_x |\xi^{\lambda_1}_{ik} x| \leq 2M/k.$$

Let

$$\Pi_1(X) = \{\mathfrak{d}_\alpha x;\ \xi^{\lambda_1}_{ik} x;\ \rho^{\lambda_1}_{ikl} x;\ \varphi^{\lambda_1}_{jk} x\}.$$

Proceeding in the same way further, we obtain an increasing sequence of families $\Pi_p(X)$ and their sum $\Pi(X)$, $\Pi(X) \subseteq C(X)$, whose elements, by means of the system of indices chosen by us, have already been put into one-to-one correspondence $\gamma : \Pi \to \Pi(X)$ with the elements of the system Π (forming the ring C). It is not hard to— show that always $M(X, \lambda_p) \leqslant M(\lambda_p)$ and that under our correspondence the norm does not increase. By linearity this correspondence extends to a homomorphism γ of the whole ring C onto the ring $D\Pi(X)$, whose norm is equal to 1. Consequently, γ is a continuous homomorphism of the ring C onto a regular subring $D\Pi(X)$ of the ring $C(X)$. In view of the choice of the functions $\xi^{\lambda_p}_{ik} x$, $\rho^{\lambda_p}_{ikl} x$, and $\varphi^{\lambda_p}_{jk} x$, the homomorphism γ is equal to 0 on all generators of the ideal J, whereby Theorem 1_R is essentially proved.

G. **Proof of Theorem 1_B and information from the theory of normed rings.**

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* By $\tilde{\xi}, \tilde{\rho}, \tilde{t}, \tilde{\varphi}$ we denote the images of the corresponding elements ξ, ρ, t, φ of the ring C under the natural homomorphism of the ring C onto the factor ring R.

Definition 3. By an h-ideal of a ring G we shall mean any closed ideal J of it such that the quotient ring G/J is isomorphic and isometric to the field of real numbers.

Every h-ideal is maximal; the converse is false. By θ_g we denote the function that assigns to each h-ideal J the number $\theta_J g$, where θ_J is the natural homomorphism of the ring G into the field of real numbers G/J.

Definition 4. By the **space of h-ideals** of the ring G we shall mean the set $B(G)$ of all h-ideals of the ring G, taken in the weakest of all its topologies in which all functions of the form θ_g are continuous.

The space $B(G)$ for any ring G is a bicompactum, and the canonical correspondence θ, which assigns to each element g of the ring G the function θg, is a homomorphism; the subring $\theta(G)$ is dense in $C(B(G))^*$.

Theorem 3. *The closed subrings of the ring $C(X)$ of the space X are in one-to-one correspondence with bicompact extensions of continuous images of the space X; the closed regular subrings correspond to bicompact extensions of the space X itself.*

Theorem 4. *To every nonzero continuous homomorphism γ of a ring G' into a ring G there corresponds a continuous mapping γ^*, which assigns to each h-ideal J from $B(G)$ the h-ideal $\gamma^{-1}J$ from $B(G')$; if, moreover, $\overline{\gamma G'} = G$, then γ^* is a homeomorphism.*

Proof of Theorem 1. The weight of the bicompactum $B_{n\tau} = B(R)$, where $R = R_{n\tau}$, is equal to τ, since the weight of a bicompactum B is always equal to the weight of the ring $C(B)$ (see (9)). By Theorems 1_R and 2, $\dim B_{n\tau} \leqslant n$. By Theorem 1_R, for an arbitrary space X of weight $\leqslant \tau$ and dimension $\leqslant n$ there exists a continuous homomorphism γ of the ring R onto some regular subring G of the ring $C(X)$. By Theorem 4 there also exists a homeomorphism of the space $B(G) = B(\overline{G})$ into $B_{n\tau}$. But by Theorem 3 the bicompactum $B(\overline{G})$ is an extension of the space X. Theorem 1 is proved.

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* We modify here the known theory of spaces of maximal ideals of normed rings over the field of complex numbers, due to I. M. Gelfand and G. E. Shilov (see (2)).

Note: Figure translations are in progress. See original paper for figures.

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