



---

Soviet-era science, translated into English

# Reports of the Academy of Sciences of the USSR

1964

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.71972>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

1964, Vol. 157, No. 2

**MATHEMATICS**

**KARL-PETER HADELER**

### **AN ESTIMATE FOR THE SPECTRUM OF NORMAL OPERATORS**

*(Presented by Academician P. S. Novikov on 14 II 1964)*

I. In 1963 H. Erman <sup>(5)</sup> generalized a long-known theorem <sup>(3)</sup>:

*If  $A$  is a completely continuous, symmetric operator in the space  $L_2(a, b)$  (or  $l_2, R_n$ ) and  $f_0 \in L_2(a, b)$ ,  $f_0 \neq 0$ , is an arbitrary function, then the eigenvalues of the operator  $A$  can be estimated by means of the values of the function  $g = Af_0/f_0$ .*

He succeeded in dropping certain assumptions on the representation of the operator in the form of an integral operator and on the sign of the function  $g$ .

Here we shall prove that this theorem is a special case of a general theorem for arbitrary normal operators in the space  $L_2(B)$ . We consider operator equations of the type  $Ax = \lambda Nx$  and applications, as well as an analogous theorem in a semi-ordered space.

II. Let  $H$  be one of the following spaces:  $L_2(B)$ —the class of measurable and square-summable complex functions on a measurable set  $B \subset R_n$  of positive measure, or the Hilbert space of sequences  $l_2$  ( $B = N$ ), or the complex space  $R_n$  with the ordinary Euclidean metric ( $B = \{1, 2, \dots, n\}$ ).

We shall always regard the elements of the space  $H$  as functions on  $B$ . Let  $f, h \in H$  (the representatives are fixed so that  $f(s), h(s)$  are defined for all  $s \in B$ ). We divide  $B$  into the measurable sets

$$A_1 = \{s \mid h(s) \neq 0\}, \quad A_2 = \{s \mid h(s) = 0, f(s) = 0\},$$

$$A_3 = \{s \mid h(s) = 0, f(s) \neq 0\}$$

and denote the characteristic function of the set  $A_i$  by  $\varphi_i$ ; then define a function  $g(s)$ , with values in the Riemann sphere of complex numbers  $S$ , by

$$g(s) = \begin{cases} \frac{f(s)}{h(s)}, & \text{for } s \in A_1, \\ b, & \text{for } s \in A_2, \\ \infty, & \text{for } s \in A_3. \end{cases} \quad b \in S \text{ arbitrary, but fixed,}$$

Changes of the values of  $g(s)$  on sets of measure zero are allowed; the class  $g$  will be denoted by  $\left(\frac{f}{h}\right)$  and called the ratio of  $f$  to  $h$ .

Consider the following regions  $K$  on the Riemann sphere:

$$K_1 = \{z \mid |z-a| \leq r\}, \quad K_2 = \{z \mid |z-a| \geq r\}, \quad \{K_3 = \{z \mid \operatorname{Re} e^{i\theta}(z-c) \geq 0\}.$$

Let  $A$  be a linear (closed) normal operator whose domain of definition  $D_A$  is dense in  $H$ . Then the spectrum  $\sigma(A) \neq \emptyset$ , and for  $z \in \rho(A)$  the resolvent  $R_z = (A - zE)^{-1}$  is a normal, bounded operator with dense domain of definition, and  $\|R_z\| = d_z^{-1}$ , where

$$d_z = \min_{\zeta \in \sigma(A)} |z - \zeta|.$$

**Theorem 1.** Using  $f_0 \in D_A$ ,  $f_0 \neq 0$ , form  $f_1 = Af_0$  and the ratio

$$g(s) = \left(\frac{f_1}{f_0}\right).$$

Suppose that the set of values of  $g(s)$  is situated in the circle  $K$ . Then the intersection  $K \cap \sigma(A) \neq \emptyset$ .

**Proof.** We shall assume that  $K \cap \sigma(A) = \emptyset$  and arrive at a contradiction.

1.  $K = K_1$ . Since  $A_3 = \emptyset$ ,  $b \neq \infty$ , we have

$$g(s) = f_1(s)[f_0(s) + \varphi_2(s)]^{-1} + b\varphi_2(s)$$

is a measurable and bounded function. For the bounded multiplication operator  $G$ , defined by the formula  $Gf = g(s)f(s)$ , we have  $\|G - aE\| \leq r < d_a$ . From  $Af_0 = Gf_0$  it follows that  $f_0 = R_a(G - aE)f_0$ , and (since  $\|f_0\| \neq 0$ )

$$\|f_0\| \leq \|R_a\| \cdot \|G - aE\| \cdot \|f_0\| < d_a^{-1} d_a \|f_0\|.$$

2.  $K = K_2$  and  $A$  is bounded. With the aid of the measurable, bounded function

$$\hat{g} = f_0[f_1 - af_0 + \varphi_2]^{-1} + \varphi_2(b - a)^{-1}$$

we define the multiplication operator  $\hat{G}f = \hat{g}(s)f(s)$ . From  $Af_0 = f_1$  it follows that  $\hat{G}(A - aE)f_0 = f_0$ . Since  $\|f_0\| \neq 0$ ,  $\|\hat{G}\| \leq \frac{1}{r}$ ,  $\|A - aE\| < r$ , we obtain a contradiction.

3.  $K = K_2$ ,  $A$  is unbounded. A trivial case.
4.  $K = K_3$ . From the assumption  $K \cap \sigma(A) = \emptyset$  it follows that  $C = (S - K) \cap \rho(A) \neq \emptyset$ ; on every straight line perpendicular to the boundary  $K'$  of the half-plane  $K$ , there is an open interval belonging to  $C$  and tangent to  $K'$ . Let  $\xi \in C$  and lie in such an interval. Define  $h_0 = f_1 - \xi f_0 \neq 0$ ,  $h_1 = f_0 \neq 0$ . Then  $f_1 = Af_0$ ,  $f_0 = R_\xi(f_1 - \xi f_0)$ ,  $R_\xi h_0 = h_1$ . For the relation  $\tilde{g}(s) = \left(\frac{h_1}{h_0}\right)$  we have  $\tilde{A}_1 = \tilde{A}_1 \cup \tilde{A}_3$ ,  $\tilde{A}_2 = \tilde{A}_2$ ,  $\tilde{A}_3 = \emptyset$ . Put  $\tilde{g}(s) := (b - \xi)^{-1}$ ; then  $\tilde{g}(s) = (g(s) - \xi)^{-1}$ . Let the finite disk  $\tilde{K}$  be the image of the domain  $K$  under the transformation  $w = (z - \xi)^{-1}$ . For  $\tilde{K}, \tilde{g}, R_z$  the theorem has already been proved in (1), consequently  $\tilde{K} \cap \sigma(R_z) \neq \emptyset$ ; since  $K \cap \sigma(A) = \emptyset$ , it follows that  $\tilde{K} \cap \sigma(R_z) = \{0\}$ . By a more refined argument one can show here that inside the disk  $\tilde{K}$  there are points of the spectrum  $\sigma(R_z)$ , so that  $K \cap \sigma(A) \neq \emptyset$ .

III. The theorem can be generalized to the case when the operator equation has the form  $Ax = \lambda Nx$ . Let the operator  $N$  be defined on the whole space  $H$ , bounded, symmetric, and positive definite,  $0 \notin \sigma(N)$ .  $N$  has a root  $Q$ ,  $QQ = N$ , with the same properties. We call  $\chi(N) = \|N\| \cdot \|N^{-1}\|$  the **condition number** of the operator  $N$ . (F. L. Bauer and A. S. Householder <sup>(2)</sup> introduced this concept for finite matrices.) Let  $A$  be a self-adjoint operator on  $D_A \subset H$  and  $\hat{A}$  the operator defined on  $D_{\hat{A}} = \{q \mid q = Qf, f \in D_A\}$  by  $\hat{A} = Q^{-1}AQ^{-1}$ . In the natural way we define  $\sigma_N(A) = \sigma(\hat{A})$ .

**Theorem 2.** By means of  $f \in D_A$ ,  $f \neq 0$ , form  $Af = f_1$ ,  $Nf = f_0$  and the relation  $g\left(\frac{f_1}{f_2}\right)$ . Let the set of values of the function  $g$  lie in the disk  $K$ :

$$K_1 = \{z \mid |z - a| \leq r\} \quad \text{or} \quad K_2 = \{z \mid |z - a| \geq r\}.$$

Then  $\hat{K} \cap \sigma_N(A) \neq \emptyset$ , where  $\hat{K}$  is defined as

$$\hat{K}_1 = \{z \mid |z - a| \leq r\chi(N)^{1/2}\}, \quad \text{or} \quad \hat{K}_2 = \{z \mid |z - a| \geq r\chi(N)^{-1/2}\}.$$

For finite matrices H. Baruch <sup>(1)</sup> had already obtained this result, but it can be sharpened. Let  $\mathfrak{D} = \mathfrak{D}(B)$  be the class of all measurable functions  $\delta(s)$  on  $B$  having the property

$$0 < \delta_1 \leq \delta(s) \leq \delta_2 < \infty, \quad \text{for all } s \in B,$$

where  $\delta_1, \delta_2$  depend on the function. The multiplication operators  $\Delta h = \delta(s)h(s)$  and  $\Delta^{-1}$  are bounded. In addition to the operator  $A - \lambda N$ , we shall consider  $\Delta A \Delta - \lambda \Delta N \Delta$

on  $D_\Delta = \{q \mid q = \Delta^{-1}f, f \in D_A\}$ . Since  $\sigma_{\Delta N \Delta}(\Delta A \Delta) = \sigma_N(A)$  and  $g(s) = \frac{Af}{f} = \frac{\Delta A \Delta \cdot \Delta^{-1}f}{\Delta N \Delta \cdot \Delta^{-1}f}$ , in Theorem 2 one may replace  $\chi(N)$  by the quantity

$$\bar{\chi}(N) = \inf_{\delta \in \mathcal{D}} \chi(\Delta N \Delta).$$

Obviously,  $\bar{\chi}(N) \geq 1$ . For diagonal matrices  $\bar{\chi}(N) = 1$ . For real matrices of simple type

$$N = \begin{pmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_0 \end{pmatrix}, \quad N_\nu = \begin{pmatrix} a_\nu & b_\nu \\ b_\nu & c_\nu \end{pmatrix} \quad (\nu = 1, \dots, m),$$

$N_0$  is a diagonal matrix,

$$\bar{\chi}(N) = \frac{\max_\nu (\sqrt{a_\nu c_\nu} + b_\nu)}{\min_\nu (\sqrt{a_\nu c_\nu} - c_\nu)}.$$

**IV. Applications and examples.** The definition, due to von Neumann, of a normal operator <sup>(6)</sup> makes it possible to construct examples of integral and differential operators. A closed operator is called **normal** if  $D_A = D_{A^*}$  and  $AA^* = A^*A$ . A differential expression  $l$  of order  $n$  in a domain  $B$  will be called **formally normal** if  $ll^*y$  and  $l^*ly$  exist for every  $y \in C^{2n}(B)$ , are continuous, and  $ll^*y = l^*ly$ . For example, the ordinary differential expression of second order with real coefficients

$$ly = py'' + qy' + ry$$

is formally normal if and only if

$$2p(p'' - q') + p'(q - p') = 0, \quad p(p^{IV} - q''') + q(p''' - q'') + 2r'(q - p') = 0$$

(in the self-adjoint case  $p' = q$ ). Every differential expression with constant coefficients is formally normal.

**Example 1.**

Fig. 1

Figure 1: Fig. 1

$$ly = y^{IV} + a(y''' + y') = \lambda y,$$

$$y(0) = y'(0), \quad y(\pi) = y'(\pi), \quad y''(0) = y''(\pi) = 0,$$

is a normal and (except for  $a = 0$ ) non-self-adjoint operator. The trial function  $f_0 = 1 + \sin x$  gives  $g = \sin x / (1 + \sin x)$ ; consequently, there exists an eigenvalue  $\lambda$  with  $|\lambda - 1/4| \leq 1/4$  for every  $a$ .

**Example 2.** The operator

$$Lu = -u_{xx} - u_{yy} + u_{xy} - 2u_x = \lambda u$$

with boundary condition  $u = 0$  on  $\Gamma$  in the triangle (see Fig. 1). The operator is normal. With the aid of the trial function  $f_0(x, y) = y\{\sqrt{3}(1+x) - y\}\{\sqrt{3}(1-x) - y\}$  we obtain

$$g(x, y) = \frac{6x(y-1) + 4\sqrt{3}}{y^3 - 2\sqrt{3}y^2 - 3x^2y + 3y},$$

where  $g$  is positive inside  $B$  and unbounded on  $\Gamma$ . In each half-plane  $\text{Re } e^{i\theta}(\lambda - 8.84) \geq 0$ ,  $-\pi/2 \leq \theta \leq \pi/2$ , there lies an eigenvalue. (Applying the Ritz-Galerkin method, we obtain that the eigenvalue  $\lambda_1$  of smallest modulus satisfies  $|\lambda_1| \leq 14$ .)

**Fig. 1**

**Example 3.**  $ly = -y'' + x^4y$  in the space  $L_2(-\infty, \infty)$ . Let  $L'_0$  be the operator on the set of finite twice differentiable functions; the closure  $L$  of this operator is already self-adjoint with discrete spectrum <sup>(7)</sup>. The trial function  $f_0 = e^{-\alpha x^2}$ ,  $\alpha > 0$ , gives  $g = 2\alpha - 4\alpha^2x^2 + x^4$ . Since

$$\max_{\alpha > 0} \min_x g(x) = 0.75,$$

there exists an eigenvalue  $\lambda \geq 0.75$ .

**Example 4.** The integral equation of the type  $Ay = \lambda[E + (N - E)]y$ , where  $A$  and  $N - E$  are symmetric, completely continuous operators and  $\sigma(N - E) > -1$ . Theorem 2 can be applied. Suppose we seek a number  $k$  for which the equation

$$y(x) = \frac{1}{2} \int_2^1 [\sin(xs) + k \cos(xs)] y(s) ds$$

has a nontrivial solution. Since

$$\alpha^2 = \int_0^1 \int_0^1 |\sin(xs)|^2 dx ds = \frac{1}{2}(1 - \text{Si}(2)) = 0.0985,$$

there is no solution for  $k = 0$ . Rewrite the preceding equation in the form

$$\frac{1}{2} \int_0^1 \cos(xs) y(s) ds = \frac{1}{2} \left[ y(s) - \frac{1}{2} \int_0^1 \sin(xs) y(s) ds \right].$$

The spectrum  $\sigma(N-E)$  is symmetric with respect to zero and lies on the interval  $|z| \leq \frac{1}{2}\alpha$ , i.e.  $\chi(N) \leq \frac{2+\alpha}{2-\alpha}$  and  $\chi(N)^{1/2} \leq 1.1716$ . With the trial function  $f \equiv 1$  we obtain  $g(x) = \sin x / (2x - 1 + \cos x)$ , so that finally such a  $k = k_0$  exists in the interval

$$1.791 \leq k_0 < 2.020.$$

V. Analogous properties hold for positive operators in partially ordered spaces. Let  $H$  be the real space  $C[0, 1]$  of continuous functions with the cone  $K$  of nonnegative functions. Let  $A$  be a bounded linear operator which maps the cone  $K$  into itself. Then the following theorem is true.

**Theorem 3.** Let  $f_0$  be an interior point of the cone. Form  $f_1 = Af_0 \in K$  and the ratio  $g(s) = f_1/f_0$ . Let  $0 < m \leq g(s) \leq M$  for all  $s \in [0, 1]$ . Then the spectral radius  $r$  of the operator  $A$  lies in the interval

$$m \leq r \leq M.$$

For finite matrices the theorem has long been known <sup>(4)</sup>. Passage to the complex space  $C[0, 1]$  gives no more general result. The proof uses certain properties of positive operators found by H. Schaefer <sup>(8)</sup>.

Received  
31 I 1964

## REFERENCES

- <sup>1</sup> H. Bartsch, Arch. Math., **4**, 133 (1953).
- <sup>2</sup> F. L. Bauer, A. S. Householder, Num. Math., **2**, 308 (1960).
- <sup>3</sup> L. Collatz, Math. Zs., **47**, 395 (1942).
- <sup>4</sup> L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, 2 Aufl., Leipzig,

1963.

<sup>5</sup> H. Ehrmann, Math. Zs., **82** H. 5, 403 (1963).

<sup>6</sup> J. v. Neumann, Ann. Math., **33**, 294 (1932).

<sup>7</sup> M. A. Neumark, *Lineare Differentialoperatoren*, Berlin, 1960.

<sup>8</sup> H. Schaefer, Pacific J., **10**, 1009 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*