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PHYSICS

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Abstract

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PHYSICS

P. SHURANYI

ON THE PECULIARITIES OF THE SOLUTION OF THE BETHE-SALPETER EQUATION IN THE ANGULAR-MOMENTUM PLANE

(Presented by Academician N. N. Bogolyubov, 14 IX 1963)

In a previous paper ([1]) we dealt with solutions of the Bethe–Salpeter equation in the ladder approximation in the theory of scalar mesons with interaction $\lambda\varphi^4$ (case I) and in the theory of scalar mesons interacting by means of the exchange of vector mesons (case II). There the case was studied in detail in which the particle masses and the total energy $\sqrt{s}$ are equal to zero. Let us denote the corresponding scattering amplitude and the kernel of the equation which it satisfies by $\overline{T}_l(p,\omega,p',\omega')$ and $\overline{V}_l(p,\omega,p',\omega')$. In this case the rightmost singularity of the amplitude in the l-plane is a fixed branch point of root type $1/\sqrt{l-l_0}$. Here we shall consider the case in which the particle masses and $\sqrt{s}$ are not equal to zero.

After a rotation of the integration contour in the ω-plane, the Bethe–Salpeter equation can be written in the form ([2])

$$T_l(p_0,\omega_0,p_2,\omega_2,s) = K_l(p_0,\omega_0,p_2,\omega_2)+$$

$$+\int_0^\infty dp_1 \int_{-\infty}^{\infty} d\omega_1 K_l(p_0,\omega_0,p_1,\omega_1)\frac{1}{F(p_1,\omega_1,s)}T_l(p_1,\omega_1,p_2,\omega_2,s), \qquad (1)$$

where

$$F(p,\omega,s) = \left[p^2+m^2+\left(\omega-\frac{i\sqrt{s}}{2}\right)^2\right]\left[p^2+m^2+\left(\omega+\frac{i\sqrt{s}}{2}\right)^2\right], \qquad (2a)$$

$$K_l(p_0,\omega_0,p_1,\omega_1) = \frac{\lambda^2}{8(2\pi)^5}\int_{4\mu^2}^{\infty}\sqrt{\frac{x-4\mu^2}{x}}\,Q_l\left(\frac{p_0^2+p_1^2+(\omega_0-\omega_1)^2}{2p_0p_1}\right)$$

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for the $\lambda\varphi^4$ theory, and

$$K_l(p_0,\omega_0,p_1,\omega_1,s)=\frac{2G^2\left(p_0^2+p_1^2+\omega_0^2+\omega_1^2+s/2+\mu^2/2\right)}{(2\pi)^3}\times$$

$$\times\, Q_l\left(\frac{p_0^2+p_1^2+\mu^2+(\omega_0-\omega_1)^2}{2p_0p_1}\right) \tag{2b}$$

for the theory with exchange of vector mesons. The kernels $V_l=\dfrac{1}{F}K_l$ are not Fredholm-type kernels. Nevertheless, the iterative solution of the integral equation (1) in the case $m=\mu=s=0$ (i.e., $\overline{T}_l$) exists if $\mathrm{Re}\,l>l_0$ ([1]). We shall prove that $\overline{V}_l>V_l>0$ in a certain region of the variables s,m,μ. (In what follows it is assumed that p,ω,p',ω' lie in the region of integration.) If this inequality is satisfied, then all iterations of $\overline{T}_l$ majorize the corresponding iterations of T_l, and therefore T_l cannot have singularities in the region $\mathrm{Re}\,l>l_0$ $(dV_l/dl<0)$.

In case (1) the function K_l satisfies the condition $K_l<K_l(\mu=0)=\overline{K}_l$, and for the function $1/F$ the following estimate is valid:

$$\frac{1}{F}=\frac{1}{(p^2+m^2+\omega^2-s/4)^2+\omega^2 s}<\frac{1}{(p^2+\omega^2)^2}$$

for

$$|s|<4m^2. \tag{3}$$

Therefore, in the region (3), $\overline{V}_l>V_l$.

In case II (see formula (26)) in the region $s>-\mu^2$ the kernel is positive. If

$$s>-\mu^2,\qquad 2m^2>s>-4m^2, \tag{4}$$

then the inequality

$$\frac{2p^2+2{p'}^2+2\omega^2+2\omega{a'}^2+s+\mu^2}{(p^2+\omega^2+m^2-s/4)^2+\omega^2 s}<\frac{2p^2+2{p'}^2+2\omega^2+2\omega{a'}^2+\mu^2}{(p^2+\omega^2)^2}$$

holds.

Using the integral representation of the Legendre function $Q_l(Z)$ [3]

$$Q_l(z)=\int_0^\infty d\varphi\,\left(z+\mathrm{ch}\,\varphi\sqrt{z^2-1}\right)^{-l-1},$$

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we obtain the inequality (for $l>0$):

$$2(p^2+{p'}^2+\omega^2+\omega{a'}^2)\,Q_l\left(\frac{p^2+{p'}^2+(\omega-\omega')^2}{2pp'}\right)>$$

$$>(2p^2+2{p'}^2+2\omega^2+2\omega{a'}^2+\mu^2)\,Q_l\left(\frac{p^2+{p'}^2+\mu^2+(\omega-\omega')^2}{2pp'}\right).$$

Thus, we find that in case II $\overline{V}_l>V_l$, if the inequalities (4) are satisfied.

For what follows it is convenient to introduce the kernels

$$\overline{V}_l^{+}=\overline{V}_l\times\Theta(p^2+\omega^2-1)\,\Theta({p'}^2+\omega{a'}^2-1),\tag{5}$$

$$\overline{V}_l^{-}=\overline{V}_l\times\Theta(1-p^2-\omega^2)\,\Theta(1-{p'}^2-\omega{a'}^2).$$

Since the kernels $\overline{V}_l^{+}$, $\overline{V}_l^{-}$, or $\overline{V}_l^{+-}=\overline{V}_l^{+}+\overline{V}_l^{-}$, satisfy the conditions

$$\overline{V}_l>\overline{V}_l^{+},\qquad\overline{V}_l>\overline{V}_l^{-},\qquad\overline{V}_l>\overline{V}_l^{+-},\tag{6}$$

the singularities of the corresponding solutions (or resolvents) lie in the region $\mathrm{Re}\,l\ll l_0$.

We shall prove that the operators

$$\frac{1}{1-\overline{V}_l^{+}},\qquad\frac{1}{1-\overline{V}_l^{-}},\qquad\frac{1}{1-\overline{V}_l^{+-}}$$

possess the singularity of the operator

$$\frac{1}{1-\overline{V}_l}$$

at the point $l=l_0$. For this we shall use the operator identity

$$\frac{1}{1-A}=\frac{1}{1-B}\,\frac{1}{1-(A-B)\dfrac{1}{1-B}}.\tag{7}$$

First set $A=\overline{V}_l$, $B=\overline{V}_l^{+-}$. $\overline{V}_l-\overline{V}_l^{+-}$ is a Fredholm-type kernel. We know that

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$$\frac{1}{1-\overline{V}_l} > \frac{1}{1-\overline{V}_l^{+-}} > 0,$$

if $l > l_0$. Therefore,

$$\frac{1}{1-\overline{V}_l^{+-}}$$

cannot have a pole at the point $l = l_0$. Suppose that

$$\frac{1}{1-\overline{V}_l^{+-}}$$

does not have the same singularity as

$$\frac{1}{1-\overline{V}_l}$$

at the point $l = l_0$; we shall show that this assumption ···

leads to a contradiction. The asymptotic behavior of the kernel $\dfrac{1}{1-\bar{V}_l}$ for $p^2+\omega^2,\ {p'}^2+\omega {a'}^2 \to \infty,\ 0$ has the form ($\bar{V}_l$ has been symmetrized) ([1])

$$\left\langle p,\omega \left| \frac{1}{1-\bar{V}_l} \right| p',\omega' \right\rangle \sim$$

$$\sim \frac{1}{\sqrt{(p^2+\omega^2)({p'}^2+\omega {a'}^2)}} \left[f_1\left(l, \frac{p}{\omega}, \frac{p'}{\omega'}\right) \Theta(p^2+\omega^2-{p'}^2-\omega{a'}^2) \left(\frac{p^2+\omega^2}{{p'}^2+\omega{a'}^2}\right)^{-x_1(l)/2} + \right.$$

$$\left. + f_2\left(l, \frac{p}{\omega}, \frac{p'}{\omega'}\right) \Theta({p'}^2+\omega{a'}^2-p^2-\omega^2) \left(\frac{{p'}^2+\omega{a'}^2}{p^2+\omega^2}\right)^{-x_1(l)/2} \right],$$

where $x_1(l)$ behaves as $\sim \sqrt{l-l_0}$ near the singularity at the point $l = l_0$, and f_1 and f_2 near this point behave as $\sim \dfrac{1}{\sqrt{l-l_0}}$. Then in the asymptotic region

$$\left\langle \left| \frac{1}{1-V_l^+} \right| \right\rangle$$

is majorized by this same expression, and we obtain that

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$$(\bar{V}_l - \bar{V}_l^+)\frac{1}{1-\bar{V}_l^+}$$

is a Fredholm-type kernel, regular at $l = l_0$. Since the Fredholm numerator is represented by a uniformly convergent series, it may be integrated term by term, and the integral of each term converges. Therefore the expression

$$\frac{1}{1-\bar{V}_l^+}\,\frac{1}{1-(\bar{V}_l - \bar{V}_l^+)\frac{1}{1-\bar{V}_l^+}}$$

has no singularity at the point $l = l_0$, which contradicts the fact that $\frac{1}{1-\bar{V}_l}$ has a singularity at this point.

The singularities of $\frac{1}{1-\bar{V}_l^+}$ in l coincide with the singularities of $\frac{1}{1-\bar{V}_l^-}$, since the following relations are valid:

$$\frac{1}{1-\bar{V}_l^-} = \frac{1}{1-\bar{V}_l^+} + \frac{1}{1-\bar{V}_l} - 1,$$

$$\left\langle p_1^2+\omega_1^2, \ldots \middle| \frac{1}{1-\bar{V}_l^-} \middle| p_2^2+\omega_2^2, \ldots \right\rangle =$$

$$= \frac{1}{\sqrt{(p_1^2+\omega_1^2)(p_2^2+\omega_2^2)}} \left\langle \frac{1}{p_1^2+\omega_1^2}, \ldots \middle| \frac{1}{1-\bar{V}_l^+} \middle| \frac{1}{p_2^2+\omega_2^2}, \ldots \right\rangle .$$

Let us now apply relation (7) to the kernels $\bar{V}_l^+$ and V_l in the regions (3) or (4). Since $V_l - \bar{V}_l^+$ is a Fredholm-type kernel, at the point l_0 T_l has a square-root singularity. Of course, it follows from formula (7) that $\frac{1}{1-V_l}$ may also possess poles in the l-plane, but in the regions (3) or (4) the positions of the poles l_i are restricted by the inequality $\operatorname{Re} l_i \le l_0$. Since the poles $l_i(s)$ are analytic functions of s, almost everywhere in the regions (3) or (4) the inequality $\operatorname{Re} l_i(s) < l_0$ is satisfied.

If the amplitude

$$T_l = \frac{1}{1-V_l} K_l$$

satisfies the unitarity condition in the elastic region for complex values of l, then the singularity at the point l_0 cannot be of the type $\sim \frac{1}{\sqrt{l-l_0}}$, but only of the

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type $\sim \sqrt{l-l_0}$. It is clear that our arguments, based on the identity (7), do not make it possible to determine which of these possibilities is actually realized.

The fact that $T_l, \overline{T}_l^+, \overline{T}_l^{+-}, \overline{T}_l$ have the same singularities is closely connected with Weyl' s theorem ([4]).

If a completely continuous self-adjoint operator is added to a self-adjoint operator, then the limiting spectrum of the operator does not change. Since a self-adjoint operator always has a complete system of generalized eigenvectors ([5]), the solution can formally be represented in the form:

$$T_l = \int S_l(\lambda) \frac{K_l(\lambda)}{1 - V_l(\lambda)}\, d\lambda + \sum S_l(\lambda_i) \frac{K_l(\lambda_i)}{1 - V_l(\lambda_i)}, \tag{8}$$

where $V_l(\lambda)$ and $S_l(\lambda)$ are the eigenvalues and eigenvectors of the operator V_l. If the continuous spectra of two operators (see the first term of equation (8)) coincide, then it is not surprising that their branch points also coincide. (The poles of the function T_l arise in the second term of relation (8), which gives the contribution of the discrete spectrum.)

The kernel $\overline{V}_l$ has a purely continuous spectrum, whereas the kernel V_l, generally speaking, also has a discrete spectrum. However, as we have shown, at least in the regions (3) and (4), the branch point under consideration is the extreme right singularity in the l-plane. Let us note that the square-root singularity in the l-plane was found in Sawyer' s work ([2]) by summing the leading terms of perturbation theory.

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Note: Figure translations are in progress. See original paper for figures.

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