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Abstract

Full Text

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OPEN MAPPINGS OF PROXIMITY SPACES

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In a lecture delivered at Humboldt University in Berlin ([3]), M. G. Katětov expressed the wish that mappings of proximity spaces analogous to open and closed mappings of topological spaces be defined and studied. The first step in this direction was taken by Katětov himself: for proximity spaces he defined quotient spaces, and thereby also proximally quotient mappings.*

In the present paper we introduce and study proximally open and regular (see below) mappings. The latter are analogous to the preclosed mappings proposed by Yu. M. Smirnov** (they were first studied by Thong ([2])). In addition, one more class of mappings is introduced which may also be regarded as analogous to preclosed mappings—we call them weakly open—which includes all preclosed mappings; under mappings belonging to this class, a proximity space that is the full preimage of a complete space itself turns out to be complete, provided that the full preimages of all points are closed in its completion (for example, bicompact); see Theorem 1. We also note that, with the aid of Theorem 2, we obtain a final answer to one problem of V. I. Ponomarev, namely, we find a necessary and sufficient condition for the extension $F : \beta X \to \beta Y$ of a mapping of completely regular spaces $f : X \to Y$ ($\beta X, \beta Y$ are the Čech extensions) to be open.

Definition 1. A mapping f of a proximity space P into a proximity space Q will be called **proximally open** or, briefly, **δ-open**, if from $A \Subset B \subset P$ it follows that $fA \Subset fB$.

It can be verified that uniformly open mappings of uniform spaces*** are δ-open, and δ-open mappings are open. For mappings of bicompacta these notions coincide.

Lemma 1. A mapping $f : P \to Q$ is δ-open if and only if one of the following two conditions is satisfied: a) if $fK = A$ and $A\delta B$ (in Q), then $K\delta f^{-1}B$; b) if $K\bar{\delta}L$ in P, then $fK\bar{\delta}\{q \mid f^{-1}q \subset L\}$.

Definition 2. A mapping $\varphi : P \to Q$ will be called **weakly open** if $y \in [\varphi A]$ implies $\varphi^{-1}y\delta A$.

Preclosed (and hence also δ-open) mappings, as is easy to see, are weakly open; a mapping of a normal space X onto a completely regular space Y will be preclosed if and only if the mapping of maximal proximities $\varphi : X_\beta \to Y_\beta$ is weakly open.

Finally, for brevity of formulation, we shall call a mapping $f : P \to Q$ **δ-complete** if the full preimages of all points of the space Q are closed subsets of the completion $\bar{P}$ of the space P.**** It is clear that every bicompact mapping is δ-complete.

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* We note that, despite the analogy, proximally quotient mappings need not be quotient mappings.

** A mapping $h : X \to Y$ is called **preclosed** if for every point $y \in Y$ and for every neighborhood O of the full preimage of this point, the open kernel $\mathrm{Int}\, hO$ of this neighborhood contains the point y. Here it should be noted that a regular mapping need not be preclosed.

*** For example, in the sense of ([4]).

**** For the definitions of completeness and completion for proximity spaces see ([7]). The condition that a set A of a space P be closed in the completion $\bar{P}$ can also be stated in a purely internal way (for example, A is such a closed set of the space P that the intersection of every centered $c\delta$-system ([7]) is nonempty, provided all its elements intersect A).

Theorem 1. If a mapping of a space P onto a complete space Q is weakly open and δ-complete, then P is complete.

Corollary 1. If a δ-complete mapping of a proximity space P onto a complete space Q is preclosed, then P is complete.

Corollary 2. If a mapping of a completely regular space X onto a space Y that is complete in the sense of Dieudonné is preclosed and bicompact, then the space X is complete in the sense of Dieudonné; a normal X is complete in the sense of Dieudonné if there exists a preclosed mapping of this space onto a space complete in the sense of Dieudonné, under which the preimages of all points are also complete in the sense of Dieudonné.

Definition $1'$**.** A mapping of proximity spaces will be called **almost open** if from $A \subseteq B$ it always follows that $fA \subseteq [fB]$.

Theorem 2. A mapping $f : P \to Q$ is almost open if and only if the extension $F : uP \to uQ$ to the (unique) bicompact extensions of these spaces is open.

Corollary. Let $f : X \to Y$ be a continuous mapping of completely regular spaces. The extension $F : \beta X \to \beta Y$ is open if and only if, for every pair of functionally separable sets A and B, the sets fA and $\{y \in Y \mid f^{-1}y \subset B\}$ are also functionally separable; in particular, if X and Y are normal, then F is open if and only if from $[A] \subset \mathrm{Int}\, B$ it follows that $[fA] \subset \mathrm{Int}[fB]$*.

Definition 3. A mapping $f : P \to Q$ will be called regular if, for every pair of sets $A\delta B$ that are close in Q, $f^{-1}A\delta f^{-1}B$ holds.

Definition 3′ is formulated analogously, but the sets A and B are assumed to be closed; such mappings we shall call **almost regular**.

It is obvious that open mappings are regular, and regular mappings are almost regular and weakly open.

Lemma 2. The following properties of a mapping $f : P \to Q$ of proximity spaces are equivalent: a) regularity; b) the image of any finite uniform covering is a uniform covering**; c) for any finite uniform covering $\mathfrak{U}$, the system $\{f\,\mathrm{St}(f^{-1}q, \mathfrak{U}) \mid q \in Q\}$ is a uniform covering; d) the image of a completely bounded neighborhood is a neighborhood; e) if $A \supset f^{-1}K$, then $fA \supset K$.

Properties b) and d) justify the name "regular mappings," while property e) also permits one to regard them as an analogue of preclosed mappings; nevertheless, regular mappings need not be preclosed, as the following example shows.

Example 1. Let E be the real line, and let N be the set of natural numbers. Map $E \times N$ onto E as follows: if $x < 0$, then $\psi(x, n) = x$; if $0 \leq x \leq 1/n$, then $\psi(x, n) = -x$; finally, if $x > 1/n$, then $\psi(x, n) = x - 2/n$.

If, however, the mapping is bicompact, then from its regularity preclosedness nevertheless follows.

Proposition 1. A mapping $f : P \to Q$ is regular if $f \mid R$ is regular and $fR = Q$; if f is regular and $S \subset Q$, then $f \mid f^{-1}S$ is also regular.

Theorem 3. Let the proximity space M be metrizable, and let the mapping $h : M \to Q$ be such that for every positive ε the system

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* V. I. Ponomarev first proved (6) that, under sufficiently broad assumptions concerning a mapping $f : X \to Y$ of normal spaces (namely, its perfectness), from the openness of f follows the openness of the extension $F : \beta X \to \beta Y$ (later this result was extended by him to multivalued mappings). After this A. Taimanov freed V. Ponomarev' s theorem from the condition of bicompactness of the preimages of points (for single-valued mappings), and A. Arhangel' skii again extended it to multivalued mappings (1). It is clear, incidentally, that the sufficiency of the Ponomarev–Taimanov condition also holds without the assumption of normality of X.

** Infinite uniform coverings need not pass to uniform coverings even under δ-open mappings; the same applies to arbitrary neighborhoods.

$\{hO_\varepsilon h^{-1}q \mid q \in Q\}$ is a uniform cover of the space Q. Then the proximity space Q is metrizable.

To see this, we shall slightly modify the metrizability criterion for proximity spaces ([8]), due to Yu. M. Smirnov:

A proximity space is metrizable if and only if it is, first, proper and, second, has a countable system of uniform covers separating any pair of distant sets.

Let K and L be distant in the space Q; let us star-refine, into the cover $\{Q \setminus K;\ Q \setminus L\}$, a finite uniform cover $\mathfrak{B}$; there exists n such that $\{O_{1/n}x \mid x \in M\} < h^{-1}\mathfrak{B}$; but then $\mathfrak{U}_n = \{hO_{1/n}h^{-1}q \mid q \in Q\}$ is inscribed in $\mathfrak{B}^*$ and, consequently, separates K and L. The mapping h, by Lemma 2, is regular; propriety, as we shall see below (Theorem 7), is preserved under mappings of a much wider class than regular ones.

Consequently, both mappings carrying uniform covers into uniform covers and mappings carrying neighborhoods into neighborhoods also preserve metrizability. It is of interest whether arbitrary regular (or at least δ-open) mappings preserve metrizability.

Proposition 2. *The complete preimage of a (closed) connected set under monotone (almost) regular mappings is connected.*

Theorem 4. *Let X and Y be some extensions of proximity spaces P and Q, and let a mapping $f : P \to Q$ have a δ-continuous extension $F : X \to Y$. Then, if f is regular and the complete preimage of a base of neighborhoods of any point $y \in Y$ under the mapping F is a δ-base of the set $F^{-1}y$*, **then from monotonicity** *follows the monotonicity of the mapping F****.

Corollary. *Let the mapping $f : P \to Q$ be δ-continuous and regular. If it is monotone (in the proximity, and a fortiori in the topological, sense), then its extension to the bicompact extensions $F : uP \to uQ$ is monotone.*

Proposition 3. *The projection of a product of proximity spaces***** onto any factor is regular; if this factor is a proper proximity space, then the projection onto it is δ-open.*

The projection onto a noncontinuous factor may also fail to be δ-open, as the following example shows.

Example 2. Let E be the real line with the usual ("metric") proximity. Consider the space $(E \cdot E) \times E$******, and in it the diagonal Δ of the first–seventh octants; its unit neighborhood O (in the sense of the usual metric of the space E^3) will be its δ-neighborhood. Nevertheless, the projection of the set O onto the first factor will not be a δ-neighborhood of the diagonal in $E \cdot E$.

Definition 4 (M. Katětov). A mapping $k : P \to Q$ is called **proximity-factorial** if Q is the strongest of all proximities on the set Q for which the mapping k is δ-continuous.

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* In the sense of ([5]).

** This condition is weaker than simple closedness: such are, for example, the projections of proximity products onto their factors.

*** Monotonicity is understood, of course, in the proximity sense, i.e. all $f^{-1}q$ are connected proximity spaces. The space P is connected if from $P = A \cup B$

and $A \delta B$ it follows that $A = \emptyset$ or $B = \emptyset$.

**** This proposition is an analogue of the well-known theorem from general topology first proved by A. D. Taimanov ([9]) for Čech extensions; the mapping f was assumed open or closed; in the paper ([2]) Dinh Nhè Thong, assuming f only preclosed, extended this theorem to arbitrary extensions (not necessarily bicompact ones, but the extension of the range must be perfect), but serious restrictions had to be imposed on the mapping F (namely, perfection; in A. D. Taimanov' s formulation the mapping F was automatically perfect), from which it has not been possible to free oneself completely here either.

***** In the sense of ([5]), Definition 1.

****** The cross is the product, and the dot is the weak product of proximity spaces (see ([5])).

Definition 4′. A mapping will be called **almost factor**, if in the preceding definition Q is assumed to be the strongest only among its homeomorphic proximities.

Theorem 5. *Every δ-continuous mapping is uniquely decomposed into a superposition of an (almost) factor mapping and a compaction (homeomorphism)*.*

Theorem 6. *If $g : P \to Y$ is a perfect mapping of a proximity space onto a completely regular topological space, then with the topology of Y there coincides a unique proximity Q, with respect to which this mapping is almost factor.*

Theorem 7. *Almost factor mappings preserve regularity.*

Indeed, let $f : P \to Q$ be almost factor, and let P be regular. The correction mapping** $f : P! \to Q!$ is δ-continuous; since $P! = P$, by the property of almost factor mappings, $Q! < Q$, i.e. $Q! = Q$.

Theorem 8. *A mapping is hereditary (almost) factor with respect to (closed) inverse*** sets if and only if it is (almost) regular****.*

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* The assertion concerning factor mappings follows from M. Katětov' s theorem (3) on the existence of a factor space for any partition.

** Work (5), Definition 3 and Theorem 8.

*** That is, with respect to sets of the form $A = f^{-1}K$.

**** For topological spaces an analogous theorem was recently proved by A. Arkhangel' skii.

Note: Figure translations are in progress. See original paper for figures.