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Abstract

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ON THE CLASSICAL SOLVABILITY OF DIFFRACTION PROBLEMS FOR EQUATIONS OF ELLIPTIC AND PARABOLIC TYPES

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Properly speaking, diffraction problems are usually understood to mean boundary-value or initial-boundary-value problems for equations of various types in the presence of two or more heterogeneous media. On the interface of these media certain matching conditions must be satisfied. For equations of second order there are two such conditions. Most often these are: continuity of the desired solution and continuity of its derivative along the conormal to the interface surface. In note ([1]) it was shown that problems of this type can be reduced by a simple device to problems of finding generalized solutions of ordinary boundary-value and initial-boundary-value problems, for which various methods of solution have been developed (including the finite-difference method). In this way the solvability of diffraction problems for equations of the principal types was proved in a generalized formulation under very weak restrictions on all the data of the problem.

Moreover, in the works ([1], [3], [4]) there were described (see also ([5])) methods for studying the improvement of the differential properties of these solutions as the differential properties of the coefficients and free terms of the equations, the interface surfaces, etc. are improved. However, in ([1]) it was noted that the results obtained in this way for equations of elliptic and parabolic types are obviously rough. For example, the classical character of a generalized solution was obtained under the condition that the coefficients of the equations outside the interface surface and the functions determining these surfaces must have generalized derivatives of order n.

Later, in the works ([2], [6]) and others, new methods were developed for studying the differential properties of generalized solutions, which led to the establishment of optimally sharp dependencies of the differential properties of generalized solutions of equations of elliptic and parabolic types on the differential properties of the coefficients of the equations. In particular, these methods, as applied to diffraction problems, make it possible to prove their classical solvability only under those assumptions on the data of the problem which are dictated by the nature of the matter.

Let us show how this is done using the following problems as an example.

Problem 1. Find a function $u(x)$, satisfying in the domain Ω the equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+a_i\frac{\partial u}{\partial x_i}+au=f(x), \tag{1}$$

$$\nu\sum_{i=1}^{n}\xi_i^2\le a_{ij}\xi_i\xi_j\le\mu\sum_{i=1}^{n}\xi_i^2,\qquad \nu,\mu=\text{const}>0, \tag{2}$$

and the following conditions on the boundary S and on the surfaces Γ of discontinuity of the first kind for the coefficients $a_{ij}(x)$:

$$u\big|_S=0;\qquad [u]_\Gamma=0;\qquad \left[a_{ij}\frac{\partial u}{\partial x_j}\cos(\mathbf{n},x_i)\right]_\Gamma=0, \tag{3}$$

where the symbol $[v]$ denotes the jump of the function v on Γ, and $\mathbf{n}$ is the normal to the surface Γ.

The surface Γ divides Ω into several domains:

$$\Omega=\Omega_1\cup\Omega_2\cup\cdots\cup\Omega_N,$$

and that part of Γ which is common to Ω_i and Ω_j will be denoted by Γ_{ij}. We assume that the boundaries S and Γ_{ij} belong to the class W_q^2 $(q>n)$.

Problem 2. Suppose that the coefficients $a_{ij}(x,t)$ have discontinuities of the first kind on the surfaces Γ'_{ij}, which, for simplicity, we assume to be cylindrical: $\Gamma'_{ij}=\Gamma'_{ij}\times[0,T]$. The problem consists in finding a solution $u(x,t)$ of the equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)+a_i\frac{\partial u}{\partial x_i}+au=f(x,t) \tag{4}$$

in the cylinder $Q_T=(\Omega\times[0,T])$ under the conditions

$$\nu_1\sum_{i=1}^{n}\xi_i^2\le a_{ij}(x,t)\xi_i\xi_j\le\mu_1\sum_{i=1}^{n}\xi_i^2;\qquad \nu_1,\mu_1=\text{const}>0; \tag{5}$$

$$u\big|_{t=0}=\varphi(x); \tag{6}$$

$$u\big|_{S'} = 0; \qquad [u]\big|_{\Gamma'} = 0; \qquad \left[a_{ij}\frac{\partial u}{\partial x_j}\cos(n, x_i)\right]\Big|_{\Gamma'} = 0, \tag{7}$$

where $S' = S \times [0, T]$ and $\Gamma' = \Gamma \times [0, T]$.

Theorem 1. Suppose that the coefficients (1) and the right-hand side $f(x)$ satisfy the conditions:

$$\left\|\frac{\partial a_{ij}}{\partial x_k};\ a_i;\ a;\ f\right\|_{L_q(\Omega_l)} \le M_1, \tag{8}$$

$$i, j, k = 1, \dots, n; \qquad l = 1, \dots, N; \qquad q > n; \qquad M_1 = \text{const} > 0.$$

Then the generalized solution $u(x)$ of problem (1), (3) from $W_2^1(\Omega)$ belongs to the classes $C_{0,\alpha}(\overline{\Omega}) \cap W_2^2(\Omega_k)$, $k = 1, 2, \dots, N$ (see ([2])), and has in Ω_k first-order derivatives Hölder-continuous up to Γ_{ij} and S, with the exception of possible junction points of two or a larger number of surfaces.

If, moreover, the coefficients (1) and $f(x)$ are smoother, namely: a_{ij}, $\partial a_{ij}/\partial x_k$, a_i, a, $f \in C_{0,\beta}(\Omega_l)$, $l = 1, \dots, N$, then the solution $u(x)$ belongs to the class $C_{2,\beta}(\Omega_k)$, $k = 1, \dots, N$, and satisfies all conditions of the problem in the form (1), (3).

For the solution of problem (4)–(7), Theorem 2 has been established.

Theorem 2. Let the coefficients and the free term of equation (4) satisfy conditions (5), (8) with constants independent of t from $[0, T]$, and, in addition,

$$\left\|\left(\frac{\partial a_{ij}}{\partial t}\right)^2;\ \frac{\partial a_i}{\partial t};\ \frac{\partial a}{\partial t};\ \frac{\partial f}{\partial t}\right\|_{L_q(Q_T^k)} \le M_2, \tag{9}$$

$$M_2 = \text{const} > 0; \qquad Q_T^k = \Omega_k \times [0, T]; \qquad k = 1, 2, \dots, N; \qquad q > n;$$

$$|\varphi|_{C_{0,\alpha}(\overline{\Omega})} < M_3, \qquad M_3 = \text{const} > 0; \qquad \varphi(x)\big|_{x \in S} = 0. \tag{10}$$

Then the generalized solution $u(x, t)$ of problem (4), (6), (7) belongs to the class $C_{\alpha,\alpha/2}(\overline{Q}_T)$ ([2]), has in $Q_{T,\varepsilon} = \Omega \times [\varepsilon, T]$ (ε is any positive number) a first derivative with respect to t, summable over Ω with any finite degree, and for any t from $[\varepsilon, T]$ is a function of class $W_2^2(\Omega_k)$, $k = 1, 2, \dots, N$. The derivatives u_{x_i} are Hölder-continuous in $Q_{T,\varepsilon}$ up to the boundary S' and Γ' with exponents β in x and $\beta/2$ in t, with the exception of possible points of intersection of two or more surfaces.

If the coefficients a_{ij}, a_i, a, f, $\partial Q_{ij}/\partial x_l$ for $(x,t) \in \overline{Q}_T^k$ $(k = 1, 2, \dots, N)$ are continuous functions satisfying in x and t the Hölder condition with exponents β_1, $\beta_1/2$, respectively, then $u(x,t) \in C_{2,1}^{\beta_1,\beta_1/2}(Q_T^k)$ $(k = 1, 2, \dots, N)$ and satisfies conditions (7) in the ordinary sense.

Using the procedure of smoothing the coefficients, the proof of Theorems 1 and 2 can be reduced to obtaining a priori estimates for classical solutions. For interior subdomains $\Omega_k' \Subset \Omega_k$, all necessary estimates have been established in ([2],[6]). Therefore it suffices to obtain them only in a neighborhood of the discontinuity surfaces !? Γ.

In Theorem 1 we first establish estimates of the integrals

$$\int_{K_\rho} |\nabla u|^{2r}\, dx \le c(r), \tag{11}$$

where the constant $c(r)$ depends on the quantities ν, μ, q, M_1 from (2), (8) and tends to infinity as $r \to \infty$; K_ρ is a ball of small radius ρ with center on Γ_{ij}. With their aid, from the equation we derive that the derivatives $\pm u_{x_s}$ in the directions tangent to Γ_{ij} satisfy the inequalities

$$\nu \int_{A_{k,\rho}} |\nabla u_{x_s}|^2 \zeta^2\, dx \le \gamma \left(\int_{A_{k,\rho}} (u_{x_s} - k)^2 |\nabla \zeta|^2\, dx + \operatorname{mes}^{1-2/q_1} A_{k,\rho} \right), \tag{12}$$

where $\gamma = \text{const} > 0$; $2/q_1 < 2/n$; $k > 0$ is an arbitrary constant; $A_{k,\rho}$ is the set of points x in K_ρ at which $u_{x_s} > k$ (or $-u_{x_s} > k$); ζ is a smooth function equal to zero outside K_ρ. This, as proved in ([2],[6]), makes it possible to obtain an estimate of the Hölder constants of u_{x_i}, $i = 1, 2, \dots, n$.

In the parabolic case we first derive the inequality

$$\int_{K_\rho} |u_t(x,t)|^r\, dx + \int_\varepsilon^t \int_{K_\rho} |\nabla u|^{2r}\, dx\, dt \le C_2(r), \tag{13}$$

where $C_2(r)$ depends on the constants ν_1, μ_1 and q, M_1 and does not depend on $t \in [\varepsilon, T]$; $C_2(r)$ tends to infinity as $r \to +\infty$. Then we obtain an estimate of the type (10), and further from equation (4) and the inequality

$$\frac{\partial}{\partial t} \int_{A_{k,\rho}(t)} (u_{x_s} - k)^2 \zeta^2\, dx + \nu_1 \int_{A_{k,\rho}(t)} |\nabla u_{x_s}|^2 \zeta^2\, dx \le$$

$$\le \gamma_1 \left[\int_{A_{k,\rho}(t)} (u_{x_s} - k)^2 |\nabla \zeta|^2\, dx + \operatorname{mes}^{1-2/q_2} A_{k,\rho}(t) \right], \tag{14}$$

where $2/q_2 < 2/n$; $\gamma_1 = \text{const} > 0$; $k > 0$ is an arbitrary number; ζ is a smooth function in $K_\rho \times [0, T]$, equal to zero outside K_ρ and for $t = 0$. From (14), as shown in [2], follows the estimate $|u_{x_i}|_{C^{\alpha,\alpha/2}(\overline{Q}^k_{T,\varepsilon})}$.

A proposition close to Theorem 1 was proved in the work [8], with the use of the subtle and laborious investigations of Giraud in potential theory.

The method of investigation set forth here is applicable without any substantial changes to several more general cases. For example, when: 1) conditions (3), (6) are nonhomogeneous; 2) the second of the conditions on Γ has the form

$$[ba_{ij} \cos(n, x_i) u_{x_j}]_\Gamma = \psi(x, t),$$

where b is a positive function in $\overline{\Omega}$, belonging to $W^2_q(\Omega_k)$, $k = 1, \ldots, n$, and having a discontinuity of the first kind on Γ; 3) the first boundary condition on S is replaced by the second or the third; 4) the interface changes smoothly with time. It is also applicable to elliptic and parabolic systems of the type considered in the work [7]. As for equations and systems of hyperbolic type, the methods for investigating them given in [1,9] lead to results that are essentially unimprovable.

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Note: Figure translations are in progress. See original paper for figures.