



---

Soviet-era science, translated into English

# THEORY OF ELASTICITY

A. Ya. ALEKSANDROV, Yu. I. SOLOV' EV

1964

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.70829>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Fig. 1

Figure 1: Fig. 1

**Abstract**

**Full Text**

## THEORY OF ELASTICITY

A. Ya. ALEKSANDROV, Yu. I. SOLOV' EV

### ON A GENERALIZATION OF THE METHOD FOR SOLVING AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY BY MEANS OF ANALYTIC FUNCTIONS TO SPATIAL PROBLEMS WITHOUT AXIAL SYMMETRY

*(Presented by Academician Yu. N. Rabotnov, 5 VIII 1963)*

The method for solving spatial axisymmetric problems of the theory of elasticity by means of relations between axisymmetric and auxiliary plane states<sup>(1-4)</sup> is extended in the present paper to the solution of one class of spatial problems without axial symmetry.

1°. Let loads be applied to an infinite plate, distributed uniformly along the axis  $Oy$  and directed parallel to the axes  $Oz$ ,  $Ox$ ,  $Oy$ . The loads  $q_{\parallel}(x)$  and  $p_{\parallel}(x)$  cause plane deformation of the plate in the plane  $zOx$ , while the load  $t_{\parallel}(x)$  causes deplanation in the direction of the axis  $Oy$  (Fig. 1a).

**Fig. 1**

Rotate these loads about the axis  $Oz$  through an angle  $\alpha$ , simultaneously changing their magnitude by multiplying the expressions  $q_{\parallel}$ ,  $p_{\parallel}$ , and  $t_{\parallel}$  by certain functions  $f_q(\alpha)$ ,  $f_p(\alpha)$ , and  $f_t(\alpha)$ . In this case the loads applied at the point  $M_1(r, \alpha + \theta)$  will pass to the point  $M(r, \theta)$ . We shall vary the angle  $\alpha$  from zero to  $2\pi$  and at the same time superpose the loads. As a result we obtain loading of the plate by certain loads  $q$ ,  $p$ , and  $t$ , directed respectively in the axial, radial, and tangential directions:

$$q(r, \theta) = \int_0^{2\pi} q_{\parallel}(x) f_q(\alpha) d\alpha \quad (x = r \cos(\alpha + \theta)),$$

$$p(r, \theta) = \int_0^{2\pi} p_{\parallel}(x) \cos(\alpha + \theta) f_p(\alpha) d\alpha + \int_0^{2\pi} t_{\parallel}(x) \sin(\alpha + \theta) f_t(\alpha) d\alpha, \quad (1)$$

$$t(r, \theta) = - \int_0^{2\pi} p_{\parallel}(x) \sin(\alpha + \theta) f_p(\alpha) d\alpha + \int_0^{2\pi} t_{\parallel}(x) \cos(\alpha + \theta) f_t(\alpha) d\alpha.$$

Expressions (1) can be used to find such types of loads of the auxiliary state and functions  $f$  that correspond to the prescribed expressions of the spatial loads. To solve this problem in the general case it is necessary to consider a series of auxiliary loads and the functions  $f$  corresponding to them, or to assume that these loads are defined by expressions into which  $\alpha$  enters as a parameter ( $q_{\parallel} = q_{\parallel}(x, \alpha)$ ,  $p_{\parallel} = p_{\parallel}(x, \alpha)$ ,  $t_{\parallel} = t_{\parallel}(x, \alpha)$ ).

We shall express the stresses and strains in the plate arising under the action of spatial loads through the stresses and strains arising under the action of the loads of the auxiliary state:

$$\begin{aligned} \sigma_r &= \int_0^{2\pi} [\sigma_{x\parallel} \cos^2(\alpha + \theta) + \sigma_{y\parallel} \sin^2(\alpha + \theta) + \tau_{xy\parallel} \sin 2(\alpha + \theta)] f(\alpha) d\alpha, \\ \sigma_{\theta} &= \int_0^{2\pi} [\sigma_{x\parallel} \sin^2(\alpha + \theta) + \sigma_{y\parallel} \cos^2(\alpha + \theta) - \tau_{xy\parallel} \sin 2(\alpha + \theta)] f(\alpha) d\alpha, \\ \tau_{r\theta} &= \int_0^{2\pi} \left[ -\frac{1}{2}(\sigma_{x\parallel} + \sigma_{y\parallel}) \sin 2(\alpha + \theta) + \tau_{xy\parallel} \cos 2(\alpha + \theta) \right] f(\alpha) d\alpha, \quad (2) \end{aligned}$$

$$\tau_{zr} = \int_0^{2\pi} [\tau_{zx\parallel} \cos(\alpha + \theta) + \tau_{zy\parallel} \sin(\alpha + \theta)] f(\alpha) d\alpha, \quad \sigma_z = \int_0^{2\pi} \sigma_{z\parallel} f(\alpha) d\alpha,$$

$$\tau_{z\theta} = \int_0^{2\pi} [-\tau_{zx\parallel} \sin(\alpha + \theta) + \tau_{zy\parallel} \cos(\alpha + \theta)] f(\alpha) d\alpha,$$

$$w = \int_0^{2\pi} w_{\parallel} f(\alpha) d\alpha, \quad u = \int_0^{2\pi} [u_{\parallel} \cos(\alpha + \theta) + v_{\parallel} \sin(\alpha + \theta)] f(\alpha) d\alpha,$$

$$v = \int_0^{2\pi} [-u_{\parallel} \sin(\alpha + \theta) + v_{\parallel} \cos(\alpha + \theta)] f(\alpha) d\alpha.$$

Here  $w, u, v$  are the axial, radial, and tangential displacements of the spatial state;  $w_{\parallel}, u_{\parallel}, v_{\parallel}$  are the displacements in the directions of the axes  $Oz, Ox, Oy$

Fig. 2

Figure 2: Fig. 2

of the auxiliary state. The stresses  $\sigma_{x\parallel}, \sigma_{y\parallel}, \sigma_{z\parallel}, \tau_{zx\parallel}$  correspond to plane strain, and  $\tau_{xy\parallel}, \tau_{zy\parallel}$  to warping. In the case of an isotropic plate  $\sigma_{y\parallel} = \nu(\sigma_{z\parallel} + \sigma_{x\parallel})$ , where  $\nu$  is Poisson's ratio.

2°. The relations between the spatial and auxiliary states can also be obtained in another way. Suppose loads  $q(r, \theta), p(r, \theta), t(r, \theta)$  are applied to an infinite plate. Let us superpose these loads while shifting them from  $-\infty$  to  $\infty$  along the axis  $O\eta$ , which makes an angle  $\alpha$  with the axis  $Oy$  (Fig. 1). With such a superposition we obtain an auxiliary state with loads defined by the expressions:

$$q_{\parallel}(\xi, \alpha) = \int_{-\infty}^{\infty} q(r, \theta) d\eta \quad \left( r = \sqrt{\xi^2 + \eta^2}, \theta = \alpha + \arccos \frac{\xi}{\sqrt{\xi^2 + \eta^2}} \right);$$

$$p_{\parallel}(\xi, \alpha) = \int_{-\infty}^{\infty} [p(r, \theta) \cos(\theta - \alpha) - t(r, \theta) \sin(\alpha - \theta)] d\eta; \quad (3)$$

$$t_{\parallel}(\xi, \alpha) = \int_{-\infty}^{\infty} [p(r, \theta) \sin(\theta - \alpha) + t(r, \theta) \cos(\theta - \alpha)] d\eta.$$

The loads  $q_{\parallel}$  and  $p_{\parallel}$  cause plane deformation of the plate in the plane  $zO\xi$ , while  $t_{\parallel}$  causes antiplane deformation in the direction of the axis  $O\eta$ .

In an analogous way one can relate the stresses and deformations arising in the plate under the action of spatial loads to the stresses and deformations corresponding to an auxiliary state of the plate. For example, for the displacements:

$$w_{\parallel}(z, \xi, \alpha) = \int_{-\infty}^{\infty} w d\eta; \quad u_{\parallel}(z, \xi, \alpha) = \int_{-\infty}^{\infty} [u \cos(\theta - \alpha) - v \sin(\theta - \alpha)] d\eta;$$

$$v_{\parallel}(z, \xi, \alpha) = \int_{-\infty}^{\infty} [u \sin(\theta - \alpha) + v \cos(\theta - \alpha)] d\eta. \quad (4)$$

Here  $w_{\parallel}, u_{\parallel}$  correspond to plane deformation, and  $v_{\parallel}$  to antiplane deformation.

3°. With the aid of the superpositions used in §§ 1° and 2°, and the techniques proposed in papers (1–4), one can establish relations between spatial and auxiliary states for a body of revolution.

**Fig. 2**

Let a certain cylinder with generator parallel to the axis  $Oy$  be in a stressed state formed by superposition of a plane deformed state ( $\tau_{yz} = \tau_{xz} = v = 0$ ) and a state corresponding to antiplane deformation of the transverse sections of the cylinder in the direction of the axis  $Oy$  ( $\sigma_y = \sigma_x = \sigma_z = \tau_{xz} = w = u = 0$ ) (Fig. 2a). By rotating the contour of the transverse section of the cylinder about the axis  $Oz$ , we cut out from the cylinder a body of revolution (for the possibility of this see (2-4)). For this body we perform a superposition of the components of stresses and displacements by rotating them relative to the body about the axis  $Oz$ , with simultaneous multiplication by a function of the angle of rotation  $f(\alpha)$ , just as was done for the plate in § 1°. Upon rotation through the angle  $2\pi$ , we obtain a spatial state of the body of revolution whose components are determined by formulas (2).

Consider a space with an axisymmetric cavity, which is in a spatial state (Fig. 2b). By moving the contour of the meridional section of the cavity along the axis  $O\eta$  from  $-\infty$  to  $\infty$ , we cut out in space a cylindrical cavity. For the space with such a cut-out cylindrical cavity one can carry out a superposition of the components of stresses and displacements by moving them relative to the space along the axis  $O\eta$  from  $-\infty$  to  $\infty$ . As a result of such superposition we obtain a stressed and deformed state of the space formed by the superposition of two states: a plane deformed state ( $\tau_{\eta z} = \tau_{\xi z} = v_\eta = 0$ ) and a state of antiplane deformation ( $\sigma_\xi = \sigma_\eta = \sigma_z = \tau_{\xi z} = w = u_\xi = 0$ ). The displacement components of this state are determined by expressions (4).

4°. In expressions (2) we introduce representations of the components of the plane deformed state and of the state of antiplane deformation through analytic functions (with the aid of the Kolosov-Muskhelishvili formulas and known

stresses for torsion). We shall assume that  $f(a) = e^{-ina}$ , and, using the properties of analytic functions, perform a number of transformations. As a result, we obtain representations of the components of the spatial state of the body of revolution in terms of integrals containing three functions  $\varphi, \psi, \Phi$ :

$$2Gw = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i} \int_{\bar{t}}^t [\chi\varphi_n(\zeta) - (2z_0 - \zeta)\varphi'_n(\zeta) - \psi_n(\zeta)] T_n \left( \frac{\zeta - z_0}{r_0 i} \right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}};$$

$$2G(u+iv) = - \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi} \int_{\bar{t}}^t [\chi\varphi_n(\zeta) + (2z_0 - \zeta)\varphi'_n(\zeta) + \psi_n(\zeta) + i\Phi_{n+1}(\zeta)] T_{n+1} \left( \frac{\zeta - z_0}{r_0 i} \right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}};$$

$$\begin{aligned} \sigma_z = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i} \int_{\bar{t}}^t [2\varphi'_n(\zeta) - (2z_0 - \zeta)\varphi''_n(\zeta) - \psi'_n(\zeta)] \times \\ \times T_n \left( \frac{\zeta - z_0}{r_0 i} \right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}; \end{aligned}$$

$$\begin{aligned} \sigma_r + \sigma_\theta &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i} \int_{\bar{t}}^t [2(1+2\nu)\varphi'_n(\zeta) + (2z_0 - \zeta)\varphi''_n(\zeta) + \\ &\quad + \psi'(\zeta)] T_n\left(\frac{\zeta - z_0}{r_0 i}\right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}; \\ \sigma_r - \sigma_\theta + 2i\tau_{r\theta} &= \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi i} \int_{\bar{t}}^t [2(1-2\nu)\varphi'_n(\zeta) + (2z_0 - \zeta)\varphi''_n(\zeta) + \\ &\quad + \psi'_n(\zeta) + i\Phi'_n(\zeta)] T_{n+2}\left(\frac{\zeta - z_0}{r_0 i}\right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}; \\ \tau_{zr} + i\tau_{z\theta} &= - \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{\pi} \int_{\bar{t}}^t [(2z_0 - \zeta)\varphi''_n(\zeta) + \psi'_n(\zeta) + \\ &\quad + \frac{i}{2}\Phi'_n(\zeta)] T_{n+1}\left(\frac{\zeta - z_0}{z_0 i}\right) \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}. \end{aligned}$$

Here  $\zeta = z + ir$ ,  $t = z_0 + ir_0$ ,  $\bar{t} = z_0 - ir_0$ ,  $\chi = 3 - 4\nu$ ,  $G$  is the shear modulus, and  $T_n\left(\frac{\zeta - z_0}{ir_0}\right)$  is a Chebyshev polynomial of the first kind. Assuming that the points  $\zeta, t, \bar{t}$  lie on the contour and introducing the representations (5) into the boundary conditions of the problem, we obtain a system of three equations for determining the boundary values of the three analytic functions.

Novosibirsk Institute  
of Railway Transport Engineers

Received  
20 VII 1963

## REFERENCES

- <sup>1</sup> A. Ya. Aleksandrov, DAN, **128**, No. 1 (1959).
- <sup>2</sup> A. Ya. Aleksandrov, DAN, **129**, No. 4 (1959).
- <sup>3</sup> A. Ya. Aleksandrov, DAN, **139**, No. 2 (1961).
- <sup>4</sup> A. Ya. Aleksandrov, *Prikl. matem. i mekh.*, **25**, issue 5 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*