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Abstract

Full Text

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BEHAVIOR OF SEMISIMPLE ALGEBRAIC GROUPS UNDER EXTENSION OF THE GROUND FIELD

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Let G be a connected semisimple algebraic group defined over a field k of characteristic 0. One says that the group G **splits over an extension** L of the field k (or that the field L splits G) if, among the maximal tori of the group G defined over L, there is a torus T that is completely split over L, i.e. T is isomorphic to a direct product of multiplicative groups of the universal domain over the field L. The group G always splits over a finite algebraic extension of the field k. An analogous notion is introduced also for Lie algebras. A semisimple Lie algebra $\mathfrak{g}$, defined over a field k, is called **split over the field** $L \supset k$ if there exists a Cartan subalgebra $\mathfrak{h}$, defined over the field L, such that for every $h \in \mathfrak{h}$ the characteristic roots of the endomorphism $Ad_{\mathfrak{g}}(h)$ lie in the field L. The following facts, formulated as lemmas, are known, and their proof can be found in works ([1],[2]).

Lemma 1. *The group G splits over L if and only if its Lie algebra splits over the field L.*

Lemma 2. *All k-forms of the semisimple Lie algebra $\mathfrak{g}$, defined over the field k, that split over the field L are isomorphic to one another over L.*

Let now k be a finite algebraic extension of the field of rational numbers. We shall be interested in the question: for which prime divisors p of the field k does the group G split over the p-adic completion k_p of the field k; in particular, for which groups is the following condition satisfied:

A. The group G splits over the field k_p for almost all p.

By Lemma 1, for this it is enough to solve the analogous problem for the Lie algebra of the group G.

Remark. It is well known (see, for example, ([3])) that a central simple algebra A over a field of algebraic numbers k splits over k_p (i.e. is isomorphic to the full matrix algebra over k_p) for almost all p. The multiplicative group of elements of the algebra A of reduced norm 1 is a semisimple algebraic group, and it satisfies condition A. The aim of the present work is to determine to what extent this regularity is preserved when passing to arbitrary semisimple groups. It turns out that in the general case this is no longer so.

Let $\mathfrak{g}$ be the Lie algebra of the group G. In the class of all k-forms of the algebra $\mathfrak{g}$, choose an algebra $\mathfrak{g}_0$ that splits over k. Denote by $A = \mathrm{Aut}(\mathfrak{g}_0)$ the group of all automorphisms of the Lie algebra $\mathfrak{g}_0$. It is known that A is an algebraic group, defined over k, which is the semidirect product $A = A^0 \cdot W$, where A^0 is the connected component of the identity (the Chevalley-Dickson group) and W is a finite group rational over k. The various k-forms of the algebra $\mathfrak{g}_0$ (up to isomorphism over k) are in one-to-one correspondence with the elements of the set $H^1(k, A_{\bar{k}})$ (4) ($\bar{k}$ denotes the algebraic closure of the field k). Let the algebra $\mathfrak{g}$ correspond to the element $h \in H^1(k, A_{\bar{k}})$. On the basis of Lemma 2, the algebra $\mathfrak{g}$ splits over k if and only if $h = 0$. The symbol 0 denotes here the neutral element of the set $H^1(k, A_{\bar{k}})$. For each prime

of a divisor p of the field k there exists a natural mapping $\varphi_p : H^1(k, A_{\bar{k}}) \to H^1(k_p, A_{\bar{k}_p})$. Our problem is equivalent to the following: for which p is $\varphi_p(h) = 0$?

Let us first consider the case when $W = (1)$, i.e., when the group $A = \mathrm{Aut}(\mathfrak{g}_0)$ is connected in the Zariski topology. Each element $h \in H^1(k, A_{\bar{k}})$ can be interpreted as a principal homogeneous space V of the connected group A over k. It is known that the variety V has a rational point over k_p for almost all p, which is equivalent to the equality $\varphi_p(h) = 0$ for almost all p. Thus, in this case the group G satisfies condition A.

We now pass to the general case. Consider the exact sequence of cohomology

$$H^1(k, A^0_{\bar{k}}) \xrightarrow{\alpha} H^1(k, A_{\bar{k}}) \xrightarrow{\beta} H^1(k, W_{\bar{k}}), \tag{1}$$

generated by the exact sequence of groups:

$$1 \to A^0_{\bar{k}} \to A_{\bar{k}} \to W_{\bar{k}} \to 1.$$

Since A is the semidirect product of its subgroups defined over k, the mapping β in (1) is surjective; further, $W_{\bar{k}} = W_k$, whence it follows that the mapping α is injective; in other words, the sequence (1) is embedded in the exact sequence

$$0 \to H^1(k, A^0_{\bar{k}}) \to H^1(k, A_{\bar{k}}) \to H^1(k, W) \to 0. \tag{2}$$

Each element of the set $H^1(k, W)$ may be regarded as a continuous homomorphism of the topological group $G(\bar{k}/k)$ into the discrete group W, up to equivalence in W. Let $G_0 \subset G$ be the kernel of this homomorphism and L the field corresponding to the group G_0 in the sense of Galois theory. The field L is a normal extension of the field k, whose Galois group is isomorphic to some subgroup of the group W. It is convenient to introduce the following definition:

Definition. We shall call a connected semisimple algebraic group G, defined over a field k, **central over** k if its Lie algebra is defined by an element

$$h \in H^1(k, A_{\bar{k}}) \quad \text{with } \beta(h) = 0.$$

(This name may be justified by the fact that, in the case when $\mathfrak{g}_0$ is of type A_n, the central group G is isogenous to the multiplicative group of elements of central simple algebra of reduced norm 1.)

Let the Lie algebra of the group G be defined by an element $h \in H^1(k, A_{\bar{k}})$, and let L be the field associated with $\beta(h)$. It is clear that the group G will be central over the field L, regarded as the field of definition of G, and moreover L is minimal with this property.

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^1(k, A^0_{\bar{k}}) & \to & H^1(k, A_{\bar{k}}) & \stackrel{\beta}{\to} & H^1(k, W) & \to & 0 \\ & & \downarrow & & \downarrow \varphi_p & & \downarrow \psi_p & & \\ 0 & \to & H^1(k_p, A^0_{\bar{k}_p}) & \to & H^1(k_p, A_{\bar{k}_p}) & \stackrel{\beta_p}{\longrightarrow} & H^1(k_p, W) & \to & 0. \end{array}$$

Let $H^1(k, W) \neq 0$, which is equivalent to the existence of a normal extension L of the field k whose Galois group is isomorphic to a nontrivial subgroup of the group W. Take $h \in H^1(k, A_{\bar{k}})$ for which $\beta(h) \neq 0$, and let L be the field associated with $\beta(h)$. The equality $\psi_p \cdot \beta(h) = 0$ is equivalent to the fact that the prime divisor p of the field k splits completely in the field L. Since $\beta_p \cdot \varphi_p(h) = \psi_p \cdot \beta(h)$, $\varphi_p(h)$ cannot be equal to 0 for almost all p; more precisely, if p does not split completely in L, then $\varphi_p(h) \neq 0$. We formulate what has been set forth in the form of a theorem.

Theorem. If a semisimple group G is central over k, then G satisfies condition A. If G is not central and L is the minimal normal field over which it becomes central, then G does not split at all points p at which the field L does not split.

As an example, let us consider groups G whose Lie algebras are k-forms of simple split Lie algebras. The complete classification of simple split Lie algebras is known; these are the algebras: $A_n,\ n \geqslant 1$; $B_n,\ n \geqslant 2$; $C_n,\ n \geqslant 3$; $D_n,\ n \geqslant 4$; E_6; E_7; E_8; F_4; G_2. Algebras of the types $A_1, B_n, C_n, E_7, E_8, G_2$ have connected automorphism groups. Hence all groups G of the indicated types are central and, consequently, satisfy condition A. For algebras of the types $A_n,\ n > 1$; $D_n,\ n > 4$; E_6, the group W has order 2; for the algebra D_4, W is the symmetric group of degree three. Thus, among groups G of these types there exist groups that do not satisfy condition A. The simplest examples of groups not satisfying condition A are the special unitary groups, which are k-forms of the groups $SL(n, \Omega)$, $n \geqslant 3$.

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Note: Figure translations are in progress. See original paper for figures.