I. A. Feldman

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Abstract

Full Text

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On the Asymptotics of Solutions of Certain Systems of Integral Equations

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In a number of problems of mathematical physics that lead to homogeneous Wiener–Hopf integral equations and to certain others, it is essential to find the asymptotics of the solutions of these equations (see, for example, ([1,2])). In the present note (Sec. 2) the asymptotics is given for solutions of systems of homogeneous integral equations of various types, in particular systems of Wiener–Hopf equations, systems of paired integral equations, and systems transposed to them. For the case of a single Wiener–Hopf equation, more complete results were obtained by M. G. Krein ([1]), Ch. 4. In Sec. 3 of the note, the results and methods of M. G. Krein are extended to the case of integral equations transposed to paired ones.

1. Let E be one of the spaces, defined on the whole axis, of complex-valued functions $L^{(p)}$ ($p \geqslant 1$), M, M^c, M^μ, C, C^0 (see ([1,3])). By $E_{(n\times 1)}$ and $E_{(n\times n)}$ we shall denote, respectively, the set of all n-dimensional vectors and the set of all square matrices of order n with elements from E. The set $E_{(n\times 1)}$ becomes a Banach space if the norm of an element $f = (f_1, f_2, \dots, f_n) \in E_{(n\times 1)}$ is defined, for example, by the equality

$$\|f\| = \sum_{j=1}^{n} \|f_j\|_E.$$

Let $\gamma(t)$ be a continuous function that does not vanish on the real axis. We shall agree to write $f(t) \in \gamma(t)E$ if $\gamma^{-1}(t)f(t) \in E$. The set $\gamma(t)E$ is a Banach space with norm

$$\|f\| = \|\gamma^{-1} f\|_E.$$

The space $\gamma(t)E_{(n\times 1)}$ and the set $\gamma(t)E_{(n\times n)}$ are defined analogously.

Everywhere below, h denotes some fixed positive number. If a matrix from $e^{-h|t|}L_{(n\times n)}$ is denoted by some lowercase letter, then the corresponding uppercase letter denotes its Fourier transform.

2. **Lemma 1.** Let the matrix-function $k(t) \in e^{-h|t|}L_{(n\times n)}$ have the property

$$\det(I - K(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty), \tag{1}$$

and let a_j $(j = 1, 2, \dots, m)$ be all the p distinct zeros of the function $\det(I - K(\lambda))$ in the strip $-h < \operatorname{Im}\lambda < h$, and p_j their multiplicities. Then the vector-functions

$$\varphi_{jq}(t) = \{e^{-ia_{jt}} P_{jq}^{(r)}(t)\}_{r=1}^{n} \qquad (j = 1, 2, \dots, m;\ q = 1, 2, \dots, p_j),$$

where $P_{jq}^{(r)}(t)$ are certain polynomials of degree $\leqslant p_j - 1$, form a basis of all solutions of the equation

$$\varphi(t) - \int_{-\infty}^{\infty} k(t-s)\varphi(s)\,ds = 0 \qquad (-\infty < t < \infty) \tag{2}$$

in any space $e^{h|t|}E_{(n\times 1)}$.*

* In the case $n = 1$, the functions $\varphi_{jq}(t) = e^{-ia_{jt}} t^{q-1}$ $(j = 1, 2, \dots, m;\ q = 1, 2, \dots, p_j)$ form a basis of all solutions of equation (2) (see ([3]), Theorem 146).

Lemma 2. Let the matrix functions $k_r(t) \in e^{-h|t|}L_{(n\times n)}$ $(r = 1, 2)$ have the property

$$\det(I - K_r(\lambda \pm ih)) \neq 0 \qquad (-\infty < \lambda < \infty;\ r = 1, 2), \tag{3}$$

and let B_1 and B_2 be linear bounded operators in the space $e^{h|t|}E_{(n\times 1)}$, mapping it into $e^{-ht}E_{(n\times 1)}$ and $e^{ht}E_{(n\times 1)}$, respectively. Then every solution $\varphi(t) \in e^{h|t|}E_{(n\times 1)}$ of the equation

$$\varphi(t) - \int_{0}^{\infty} k_1(t-s)\varphi(s)\,ds - (B_1\varphi)(t) = 0 \qquad (0 < t < \infty),$$

$$\varphi(t) - \int_{-\infty}^{0} k_2(t-s)\varphi(s)\,ds - (B_2\varphi)(t) = 0 \qquad (-\infty < t < 0) \tag{4}$$

admits the representations

$$\varphi(t) = \left\{ \sum_{j=1}^{m_1} e^{-i\alpha_j^{(1)} t} P_{jk}^{(1)}(t) \right\}_{k=1}^{n} + \varphi_1(t), \tag{5}$$

$$\varphi(t) = \left\{ \sum_{j=1}^{m_2} e^{-i\alpha_j^{(2)} t} P_{jk}^{(2)}(t) \right\}_{k=1}^{n} + \varphi_2(t), \tag{6}$$

where the vector functions $\varphi_1(t)$ and $\varphi_2(t)$ belong to the spaces $e^{-ht}E_{(n\times 1)}$ and $e^{ht}E_{(n\times 1)}$, respectively; $\alpha_j^{(r)}$ $(r=1,2;\ j=1,2,\dots,m_r)$ are all the distinct zeros of the function $\det(I-K_r(\lambda))$ in the strip $-h<\operatorname{Im}\lambda<h$, $p_j^{(r)}$ are their multiplicities, and $P_{jk}^{(r)}(t)$ are polynomials of degrees $\le p_j^{(r)}-1$.

Lemma 3. Let the matrix functions $k_{ir}(t)\in e^{-h|t|}L_{(n\times n)}$ $(i,r=1,2)$, and suppose that the conditions

$$\det(I-K_{rr}(\lambda\pm ih))\neq 0 \qquad (-\infty<\lambda<\infty;\ r=1,2) \tag{7}$$

are fulfilled. Then in each of the spaces $e^{h|t|}E_{(n\times 1)}$ the equation

$$\varphi(t)-\int_0^\infty k_{11}(t-s)\varphi(s)\,ds-\int_{-\infty}^0 k_{12}(t-s)\varphi(s)\,ds=0 \qquad (0<t<\infty),$$

$$\varphi(t)-\int_0^\infty k_{21}(t-s)\varphi(s)\,ds-\int_{-\infty}^0 k_{22}(t-s)\varphi(s)\,ds=0 \qquad (-\infty<t<0) \tag{8}$$

has the same and only the same solutions.

Lemma 3 is proved essentially analogously to Theorem 5.1 in [(4)]. With the aid of the lemmas given above, the following is established.

Theorem 1. Let the matrix functions $k_{ir}(t)\in e^{-h|t|}L_{n\times n}$ $(i,r=1,2)$, and suppose that conditions (7) are fulfilled. Then every solution $\varphi(t)\in e^{h|t|}E_{(n\times 1)}$ of equation (8) admits the asymptotic expansions

$$\varphi(t)=\left\{\sum_{j=1}^{m_1} e^{-i\alpha_j^{(1)}t}P_{jk}^{(1)}(t)\right\}_{k=1}^{n}+o(e^{-ht}) \qquad (t\to\infty),$$

$$\varphi(t)=\left\{\sum_{j=1}^{m_2} e^{-i\alpha_j^{(2)}t}P_{jk}^{(2)}(t)\right\}_{k=1}^{n}+o(e^{ht}) \qquad (t\to-\infty), \tag{9}$$

where $\alpha_j^{(r)}$ $(r=1,2;\ j=1,2,\dots,m_r)$ are all the distinct zeros of the function $\det(I-K_{rr}(\lambda))$ in the strip $-h<\operatorname{Im}\lambda<h$; $p_j^{(r)}$ are their multiplicities and $P_{jk}^{(r)}(t)$ are polynomials of degrees $\le p_j^{(r)}-1$.*

From Theorem 1 the following propositions follow.

Corollary 1. Let the matrix function $k(t)\in e^{-h|t|}L_{(n\times n)}$ satisfy condition (1). Then every solution $\varphi(t)\in e^{ht}E^+_{(n\times 1)}$** of the Wiener–Hopf system of equations

$$\varphi(t) - \int_0^\infty k(t-s)\varphi(s)\,ds = 0 \qquad (0 < t < \infty)$$

admits the asymptotic expansion

$$\varphi(t) = \left\{ \sum_{j=1}^{m} e^{-i\alpha_j t} P_{jk}(t) \right\}_{k=1}^{n} + o(e^{-ht}) \qquad (t \to \infty),$$

where α_j $(j = 1, 2, \dots, m)$ are all the distinct zeros of the function $\det(I - K(\lambda))$ in the strip $-h < \operatorname{Im}\lambda < h$, p_j are their multiplicities, and $P_{jk}(t)$ are polynomials of degrees $\leq p_j - 1$.

Corollary 2. Let the matrix functions $k_r(t) \in e^{-h|t|}L_{(n\times n)}$ satisfy conditions (3). Then every solution $\varphi(t) \in e^{h|t|}E_{(n\times 1)}$ of the system of paired integral equations

$$\varphi(t) - \int_{-\infty}^\infty k_1(t-s)\varphi(s)\,ds = 0 \qquad (0 < t < \infty),$$

$$\varphi(t) - \int_{-\infty}^\infty k_2(t-s)\varphi(s)\,ds = 0 \qquad (-\infty < t < 0)$$

and of the transposed system

$$\varphi(t) - \int_0^\infty k_1(t-s)\varphi(s)\,ds - \int_{-\infty}^0 k_2(t-s)\varphi(s)\,ds = 0 \qquad (-\infty < t < \infty) \quad (10)$$

admits the asymptotic expansions (9), where $\alpha_j^{(r)}$ $(r = 1, 2;\ j = 1, 2, \dots, m_r)$ are all the distinct zeros of the function $\det(I - K_r(\lambda))$ in the strip $-h < \operatorname{Im}\lambda < h$, $p_j^{(r)}$ are their multiplicities, and $P_{jk}^{(r)}(t)$ are polynomials of degrees $\leq p_j^{(r)} - 1$.

3. We now consider the equation transposed to the paired integral equation (10) in the space $e^{h|t|}E$. With respect to the functions $k_1(t)$ and $k_2(t)$ we shall assume that they belong to the space $e^{-h|t|}L$ and possess the property

$$1 - K_r(\lambda \pm ih) \neq 0 \qquad (-\infty < \lambda < \infty;\ r = 1, 2). \quad (11)$$

Equation (10) can be reduced to the equivalent Hilbert boundary-value problem for a contour consisting of a pair of parallel straight lines ([6]). With the aid of the reduction to the boundary-value problem and the results of the paper

([7]), equation (10) can be effectively solved in the space $e^{h|t|}E$. Introduce the notation:

$$\nu_1 = \operatorname{ind}(1 - K_1(\lambda + ih)), \qquad \nu_2 = \operatorname{ind}(1 - K_2(\lambda - ih)).$$

Equation (10) has in any space $e^{h|t|}E$ exactly $\chi = \nu_2 - \nu_1$ linearly independent solutions if $\chi > 0$, and has only the zero solution if $\chi \leq 0$ (see ([6],[7])).

* Asymptotic expansions of this kind are also valid for solutions of the homogeneous equations (1) and (2) from ([5]), if by $k_{ij}(t)$ and $k_j(t)$ $(i, j = 1, 2)$ one understands matrix functions from $e^{-h|t|}L_{(n\times n)}$.

** For the definition of E^+, see ([4]).

In this section we shall refine the asymptotic expansions (9) for the functions of a certain basis of all solutions of equation (10).

Let $1 - K_1(\lambda)\alpha_j$ $(j = 1, 2, \dots, n)$ denote all distinct zeros of the function in the strip $-h < \operatorname{Im}\lambda < h$, numbered so that

$$\operatorname{Im}\alpha_1 \geq \operatorname{Im}\alpha_2 \geq \dots \geq \operatorname{Im}\alpha_n,$$

and let p_j denote their multiplicities.

Theorem 2. Suppose that the functions $k_j(t)$ $(j = 1, 2)$ satisfy conditions (11), and that $\varkappa = \gamma_2 - \gamma_1 > 0$. Then one can find functions $\gamma(t)$ $(0 < t < \infty)$ and

$$G(\lambda) = 1 + \int_0^\infty \gamma(t) e^{i\lambda t}\, dt$$

such that:

a) if

$$m = \sum_{j=1}^{n} p_j \leq \varkappa,$$

then $\gamma(t) \in e^{-ht}L^+$, and to each zero $\alpha = \alpha_j$ $(j = 1, 2, \dots, n)$ of the function $1 - K_1(\lambda)$ in the strip $-h < \operatorname{Im}\lambda < h$ there correspond $p = p_j$ solutions of equation (10) with the asymptotic expansion, as $t \to \infty$,

$$\varphi_r(t) = e^{-i\alpha t} \sum_{j=0}^{r} \frac{(-i)^j t^{r-j}}{j!(r-j)!} G^{(j)}(\alpha) + o(e^{-ht}) \qquad (r = 0, 1, \dots, p-1);$$

the remaining $\varkappa - m$ linearly independent solutions belong to an arbitrary space $e^{-ht}E$;

b) if $m > \varkappa$, then $\gamma(t) \in e^{ct}L^+$ $(\operatorname{Im}\alpha_{\tau+1} < c < \operatorname{Im}\alpha_\tau)$, where τ is the greatest integer such that

$$p_1 + p_2 + \dots + p_\tau \leq \varkappa$$

and $\operatorname{Im}\alpha_\tau > \operatorname{Im}\alpha_{\tau+1}$, and to each upper zero $\alpha = \alpha_j$ $(j = 1, 2, \dots, \tau)$ there correspond $p = p_j$ solutions of equation (10) with the asymptotic expansion, as $t \to \infty$,

$$\varphi_r(t) = e^{-i\alpha t}\sum_{j=0}^{r}\frac{(-i)^j t^{r-j}}{j!(r-j)!}G^{(j)}(\alpha) + \sum_{j=\tau+1}^{n} P_{jr}(t)e^{-i\alpha_j t} + o(e^{-ht})$$

$$(r = 0, 1, \dots, p-1),$$

where $P_{jr}(t)$ are polynomials of exact degree $p_j - 1$ $(j = \tau+1, \dots, n)$; the remaining $\varkappa - p_2 - \dots - p_\tau$ linearly independent solutions admit asymptotic expansions, as $t \to \infty$, of the form

$$\sum_{j=\tau+1}^{n} P_j(t)e^{-i\alpha_j t} + o(e^{-ht}),$$

where $P_j(t)$ are polynomials of degree $\le p_j - 1$ $(j = \tau+1, \dots, n)$.*

We do not give the analogue of Theorem 2 that gives the asymptotics of solutions of equation (10) as $t \to -\infty$.

All the results presented above remain valid for the discrete analogues of the corresponding equations.

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Institute of Physics and Mathematics
Academy of Sciences of the Moldavian SSR

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* For solutions of the Wiener–Hopf equation, the asymptotic expansions given above were found by M. G. Krein ([1], Ch. 4), with the difference that the remainder term in (1) has order $o(e^{-(h-\varepsilon)t})$, where ε is an arbitrary positive number.

Note: Figure translations are in progress. See original paper for figures.