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Abstract

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MATHEMATICS

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ISOTOPY OF ELEMENTS AND SPHERES IN n-DIMENSIONAL SPACE FOR $k > \frac{2}{3}n - 1$

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1. Here a brief description is given of the method of local rearrangement of homeomorphisms of n-dimensional Euclidean space E^n, some results obtained with its help are formulated, and an outline is given of the proof of one of them. The method also makes it possible to consider embeddings of polyhedra in E^n.

In E^n a Cartesian coordinate system with origin O is fixed. Let k be a fixed integer, $k < \frac{2}{3}n-1$. Denote by T the $(n-1)$-dimensional subspace $Ox_1, \dots, x_{k-1}$, by F the subspace spanned by T and the axis Ox_k, and by Y the orthogonal complement to T. By E^n+, E^n- and F_+, F_- are denoted the half-spaces, reckoning along the axis Ox_k; B^{k-1} is the unit ball in T^k, B^k in F.

An embedding $p : M^k \to E^n$ of a manifold M^k in E^n is called **locally flat** if for each point $x \in pM$ there exists a neighborhood H in E^n such that the pair $(H, H \cap pM)$ is homeomorphic to the standard pair (E^n, F) or to the pair (E^n, F_+). It is established that a locally flat embedding in E^n of a closed simplex of any dimension is topologically equivalent to the standard one. An **isotopy** in E^n is a system of homeomorphisms of E^n onto itself $g_t(x)$, continuous in the aggregate of the variables $x \in E^n$, $t \in [0,1]$; $g_0(x)$ is always assumed to be the identity. Brown and Gluck called in ([1]) a homeomorphism **stable** if it is representable as a superposition of homeomorphisms each of which is the identity on some region. From the classical result of Alexander it follows that a homeomorphism that is the identity on a region is isotopic to the identity under an isotopy that is the identity on the region. From this follows the equivalence of Theorems 1 and 1′. Theorems 2 and 3 are consequences of Theorem 1′ and Stallings' result ([2]). Embeddings p and $p' : X \to E^n$ are called **isotopic** if there exists an isotopy g_t in E^n such that $p = g_1 p'$.

Theorem 1. *If $k < \frac{2}{3}n - 1$, then every homeomorphism $h : E^n \to E^n$ is representable as a superposition of two homeomorphisms $h = h_{kh}n$, where h_n is stable, and h_k is the identity on a k-dimensional hyperplane.*

Theorem 1′. *If $k < \frac{2}{3}n - 1$, then every locally flat embedding of a k-simplex in E^n is isotopic to the standard rectilinear embedding under an isotopy of E^n that is the identity outside some bounded region.*

Theorem 2. *If $k < \frac{2}{3}n - 1$, then every locally flat embedding of a k-sphere in E^n is isotopic to the embedding of the boundary of a straight $(k+1)$-simplex under an isotopy fixed outside a bounded region.*

Theorem 3. *If $k < \frac{2}{3}n - 1$, then two locally flat embeddings of a k-simplex in E^n, coinciding on the boundary, are isotopic under an isotopy that is the identity on the image of the boundary and outside a bounded region.*

2. Horns. Consider surfaces in E^n that are given by equations of the form

$$x_k = t\sqrt{x_{k+1}^2 + \cdots + x_n^2}, \qquad -\infty \le t \le \infty.$$

For infinite values of t one obtains respectively F_- and F_+. The surface with coefficient t is the product of T and a cone in Y with vertex O, axis Ox_k, and aperture angle $\operatorname{arc\,ctg} t$. We shall call these surfaces **horns**, and denote the horn with coefficient t by R_t. If the set L is closed outside any neighborhood of B^{k-1}, then we say that L **touches** R_t, if

for every $\varepsilon > 0$ there exists $\delta > 0$ such that in the δ-neighborhood of B^{k-1} the set L is nonempty and lies between $R_{t-\varepsilon}$ and $R_{t+\varepsilon}$.

Curved horns. A surface in E^n (the topological image of an $(n-1)$-hyperplane) is called a **curved horn** $\mathfrak{R}_t$ if it is tangent to R_t, contains T, and intersects exactly once each semicircle with center on T, with diameter parallel to the axis Ox_k, and with plane orthogonal to T. We shall regard two curved horns as coinciding if they coincide in some neighborhood of B^{k-1}. The following two propositions are easily proved:

A. *If a set L, closed outside every neighborhood B^{k-1}, lies on one side of the horn R_t, then there exists a curved horn $\mathfrak{R}_t$ lying between R_t and L; and if, in addition, L touches R_t, then there exists a curved horn $\mathfrak{R}'_t$ such that L lies between $\mathfrak{R}'_t$ and R_t.*

B. *If the curved horns $\mathfrak{R}_{t_1}$ and $\mathfrak{R}_{t_4}$ do not intersect, and $\mathfrak{R}_{t_2}$ and $\mathfrak{R}_{t_3}$ lie between them, then there exists a homeomorphism of E^n onto itself, identical on T, outside a prescribed neighborhood B^{k-1}, and outside the region between $\mathfrak{R}_{t_1}$ and $\mathfrak{R}_{t_4}$, which carries $\mathfrak{R}_{t_2}$ onto $\mathfrak{R}_{t_3}$.*

3. The principal homeomorphism. It is easy to construct a homeomorphic mapping
$g : E^n \setminus B^{k-1} \to E^n \setminus B^k$ with the following property: a sequence of points then and only then converges to a point $x \in \operatorname{Int} B^k$ when its inverse image converges to the projection of x onto B^{k-1} and at the same time touches the horn R_t, where $\left|\frac{1}{t}\right|$ is equal to the radius of the sphere which in F passes through the

boundary of B^{k-1} and x, and the sign of t is equal to the sign of the abscissa of the k-th coordinate of x.

4. The principal lemma. The proof of all the results is based on this lemma. In particular, the dimensional restriction $k < {}^2/_3\, n - 1$ is connected with the fact that the lemma has not been proved for large k. The lemma is also true for $n = 3,\ k = 1$ or 2, but here a special proof is required. Thus the first unproved case is $n = 4,\ k = 2$. The proof of the lemma follows essentially the proof of the Stallings–Zeeman engulfing lemma ([3]). The difference is that the Stallings p-connectedness conditions are not fulfilled and an infinite polyhedron has to be engulfed.

Lemma. *Let Q_t be the closed region bounded by the horn R_t; let $h : Q \to E^n$ be a homeomorphic mapping, identical on T, and let $\varepsilon > 0$ and a neighborhood $H(B^{k-1})$ be given. There exists a homeomorphism $e : E^n \to E^n$, identical on $T \cup (E^n \setminus H)$, such that ehQ_t contains a ball segment which lies in $F \cap Q_t$ and is based on B^{k-1}. If $hQ_t \subset Q_t$, then one may require that $E^n \setminus Q_t$ remain fixed.*

It is convenient to take for B^{k-1} a straight simplex in T, and for the ball segment which is to be engulfed, a k-simplex B^k in $F \cap Q$, for which B^{k-1} serves as the base and the vertex is sufficiently close to B^{k-1}. Put $hQ = V$ and $hB^k = \mathfrak{B}$. Let Δ^{k+1} be the standard $(k+1)$-simplex, Δ^{k-1} its $(k-1)$-face, and Δ^k and Δ^k_1 two k-faces adjacent to Δ^{k-1}.

The proof of the lemma splits into two steps. First one constructs a continuous mapping $\varphi : \Delta^{k+1} \to E^n$, which maps Δ^k linearly onto B^k and Δ^{k-1} linearly onto B^{k-1}, and Δ^k_1 homeomorphically onto $\mathfrak{B}$, with $\varphi(\Delta^{k+1} \setminus \Delta^{k-1}) \subset E^n \setminus T$. The construction of this mapping uses only the homotopy properties of the pair (E^n, T) and can be carried out in all dimensions.

Then φ is put in general position with respect to the triangulation $\Delta^{k+1}/\Delta^{k-1}$. In doing so, φ must still map Δ^k linearly onto B^k, and $\varphi\Delta^k_1 \subset V$. In Δ^{k+1} one considers the inverse image of all pairwise intersections of the images of open simplexes. Through each of its points a segment is drawn parallel to the edge opposite Δ^{k-7}, and the image under φ of the resulting polyhedron is taken. The image is denoted by S. In view of $k < {}^2/_3\, n - 1$: a) $\dim S < n/3 + 1$; b) $E^n \setminus T$ has the homotopy type of an s-sphere for $s > n/3 + 1$.

Therefore the pair $(E^n \setminus T, V)$ is $(n/3 + 1)$-connected. Using the proof of the absorption lemma, we can construct a homeomorphism e', identical on T, and such that $e'V \supset S$. The remaining part of the image consists of piecewise linear cells, situated "regularly" with respect to $e'V$ and with respect to one another. Then again, using the absorption lemma, one constructs a homeomorphism $e''V$ such that $e''e'V \supset \varphi\Delta^{k+1}$. Put $e = e''e'$. Although here one has to apply the elementary operations of the proof of the absorption lemma an infinite number of times, it is easily shown that the process converges and gives a homeomorphism in the limit.

5. Scheme of the proof of Theorem 1. For $k = 0$ the theorem is trivial.

Suppose that it is true for dimensions smaller than k. Then we may assume that the given homeomorphism is identical on T. We shall show that there exists a homeomorphism $\bar{h}$, coinciding with h outside some neighborhood of B^{k-1} and identical on B^k. From this it is easy to obtain that there exists a homeomorphism coinciding with h on some region and identical on all of F. This is what is required. Let us first assume that h has the following property:

C. *The image of each horn* $R_t,\ |t|<1$, *touches* R_t.

In this case, as the required homeomorphism $\bar{h}$ one may take the homeomorphism ghg^{-1}, extended identically on B^k. That such an extension leads to a homeomorphism follows at once from the property of g (item 4) and condition C. Therefore it remains to show that there exists such a homeomorphism e, fixed on T and the complement of some bounded neighborhood of B^{k-1}, that eh has property C.

It is convenient to regard two copies E^n as given and h as mapping one of them onto the other. The corresponding sets are denoted in the same way, but in the preimage a bar is added above. Thus the horns in the image are denoted, as before, by R_t, and in the preimage by $\bar{R}_t$; put $h\bar{R}_t = U_t,\ h^{-1}R_t = \bar{U}_t$. The crooked horns are denoted by $\mathfrak{R}_t$ and $\bar{\mathfrak{R}}_t$. Denote by Q_t^+ the positive side of R_t, by Q_t^- the negative side; correspondingly in the preimage, $\bar{Q}_t^{\pm}$; put $h\bar{Q}_t^{\pm} = V_t^{\pm},\ h^{-1}Q_t^{\pm} = \bar{V}_t^{\pm}$. Note that if condition C is satisfied for a countable set of t' s dense in $(-1,1)$, then it is satisfied for all $t\in(-1,1)$. We construct e as an infinite superposition of homeomorphisms, achieving the fulfillment of the condition successively for all dyadic rational $t\in[-1,1]$. The successive construction of homeomorphisms is based only on the lemma and proposition B. Therefore each homeomorphism may be taken to be an ε-homeomorphism, identical outside a neighborhood of B^{k-1}, where $\varepsilon>0$ and the neighborhood are fixed for the given homeomorphism. From this it follows that the construction can be carried out so that in the limit a homeomorphism is obtained, identical on T and outside some neighborhood of B^{k-1}.

Let the t_i run in a definite order through the dyadic rational numbers in $[-1,1]$: $t_0=0,\ t_1=1,\ t_2=-1,\ t_3=1/2$, and so on. If all numbers of the form $q/2^s$ for a fixed s have already been numbered, then next, in decreasing order, the numbers $q/2^{s+1}$ with odd q are numbered. The construction is inductive. For convenience we shall assume that each homeomorphism e_a, which is constructed at some step in E^n or in $\bar{E}^n$, changes h, i.e. if e_a is constructed in $\bar{E}^n$, then henceforth we take $\bar{h}e_a^{-1}$ for h and denote by $\bar{U}_t$ $e_a h^{-1}R_t$ and by U_t $he_a^{-1}\bar{R}_t$; while if e_a is constructed in E^n, then henceforth we take $e_a h$ for h, denote $e_a h\bar{R}_t$ by U_t, and $h^{-1}e_a^{-1}R_t$ by $\bar{U}$.

Zeroth step. 1) Using the lemma, we construct a homeomorphism in E^n which carries V_0^+ into a position in which it covers a certain spherical segment D_0^+ in F_+, resting on B^{k-1}.

2) Using A and B, we first construct a crooked horn $\mathfrak{R}_\infty$ separating D_0^+ from U_0, and carry it to R_0 (i.e. to a crooked horn coinciding with R_0 in some

neighborhood of B^{k-1}).

3) Leaving Q_0^+ fixed, we make V_0^- enclose D_0^- (a ball segment in F_- supported on B^{k-1}).

4) We construct a crooked horn $\mathfrak{R}_{-\infty}$ separating U_0 from D_0^-, and then transform $\mathfrak{R}_{-\infty}$ into some crooked horn $\mathfrak{R}_0^-$ touching R_0 on the negative side. Now U_0 lies between R_0 and $\mathfrak{R}_0^-$, hence touches R_0 on the negative side. Consequently, $\bar{U}_0 \subset \bar{Q}_0^+$.

5) In the preimage, leaving $\bar{Q}_0^-$ fixed, we make $\bar{V}_0^+$ enclose $\bar{D}_0^+$ (a ball segment in $\bar{F}_+$ supported on B^{k-1}).

6) We construct a crooked horn $\bar{\mathfrak{R}}_\infty^-$ separating $\bar{D}_0^+$ from $\bar{U}_0$, and then, leaving $\bar{Q}_0^-$ fixed, transform it into a crooked horn touching $\bar{R}_0$ on the positive side. We have now achieved that U_0 touches R_0 on the negative side, while $\bar{U}_0$ touches $\bar{R}_0$ on the positive side.

Inductive passage. Suppose that we have already achieved that, for every t_i, $i < i_0$, U_{t_i} touches R_{t_i} on the negative side and, simultaneously, $\bar{U}_{t_i}$ touches $\bar{R}_{t_i}$ on the positive side.

1) Consider those t_i for which $t_i > t_{i_0}$, $i < i_0$. In the preimage, for each such t_i, we construct, according to A, a crooked horn $\bar{R}_{t_i}^+$ touching $\bar{R}_{t_i}$ on the positive side, so that $\bar{U}_{t_i}$ lies between it and $\bar{\mathfrak{R}}_{t_i}^+$. Then, with $\bar{Q}_{t_{i_0}}^-$ fixed, we transform each $\bar{\mathfrak{R}}_{t_i}^+$ into $\bar{R}_{t_i}$ and simultaneously transform $\bar{R}_{t_i}$ into a crooked horn touching $\bar{R}_{t_i}$ on the negative side. Then U_{t_i} ($t_i > t_{i_0}$, $i > i_0$) lies between R_{t_i} and the horn adjacent to it on the positive side $R_{t_i'}$, $i' < i_0$. At the same time $U_1 \subset Q_1^+$.

2) Denote by $t_{i'}$ and $t_{i''}$ the two numbers adjacent to t_{i_0} among the numbers t_i, where $i < i_0$. All U_{t_i} with $i < i_0$ lie outside the region between $R_{t_{i'}}$ and $R_{t_{i''}}$, while $U_{t_{i_0}}$ lies in it. We construct crooked horns $\mathfrak{R}'$ and $\mathfrak{R}''$ separating $U_{t_{i_0}}$, respectively, from $R_{t_{i'}}$ and $R_{t_{i''}}$, and then, with $Q_{t_{i'}}^+ \cup Q_{t_{i''}}^-$ fixed, transform $\mathfrak{R}'$ to $R_{t_{i_0}}$, and $\mathfrak{R}''$ to a crooked horn touching $\mathfrak{R}_{t_{i_0}}$ on the negative side.

3) Similarly, using the fact that each U_{t_i}, where $1 > t_i > t_{i_0}$, $i < i_0$, lies between two adjacent horns, we ensure that these U_{t_i} touch R_{t_i} on the negative side.

4) Leaving Q_1^- fixed, by the lemma we construct in $\bar{E}$ a homeomorphism as a result of which $\bar{V}_1^+$ encloses $\bar{D}_{i_0}^+$. Then we separate $\bar{U}_1$ from $\bar{D}_{t_i}^+$ by a crooked horn, which, with $\bar{Q}_{t_{i'}}^-$ fixed, we transform to R_1 so that U_1 touches R_2 on the negative side, where $t_{i'}$ is the number adjacent to 1 among the numbers t_i, $i < i_0$. Now the tangency conditions are satisfied in the image. It remains to carry out constructions analogous to the last three steps in the preimage, and then we shall be ready to

pass to t_{i_0+1}. This completes the induction. The fact that, as a result of these constructions, in the limit condition C will be satisfied for all dyadic rational numbers in $(-1, 1)$, is verified directly.

6. These results in dimensions k not exceeding $\frac{1}{2}n - 2$ can be derived from results announced by Gluck ([4]) and Greathouse ([5]). On the other hand, the method set forth here makes it possible to obtain the results of Gluck and Greathouse for $k < \frac{2}{3}n - 1$.

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Note: Figure translations are in progress. See original paper for figures.