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Abstract

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THE OBJECT OF A CENTRO-PROJECTIVE CONNECTION ON A MANIFOLD WITH AN ALMOST COMPLEX STRUCTURE

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1. Let $\mathfrak{M}_{2n}$ be a $2n$-dimensional differentiable manifold with an almost complex structure, i.e., on $\mathfrak{M}_{2n}$ there is defined a tensor F^i_j $(i, j, k, l = 1, 2, \dots, 2n)$ such that $F^i_j F^k_i = -\delta^k_j$. Suppose that on $\mathfrak{M}_{2n}$ an object of centro-projective connection of a differential neighborhood of the third order $(\Gamma^i_{jk}, \Gamma_{jk})$ and a copuncture a_i are given [1].

The object with components

$$\begin{gathered} \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + F^s_j F^i_{s,k}, \\ \tilde{\Gamma}_{jk} = \Gamma_{jk} - a_i F^s_j F^i_{s,k} + (\delta^s_j - F^s_j) D_k a_s, \end{gathered} \tag{1}$$

where $D_k a_s = \partial a_s / \partial x^k - a_p \Gamma^p_{sk} - \Gamma_{sk}$, and $F^i_{s,k}$ denotes covariant differentiation in the connection Γ^i_{jk}, defines on $\mathfrak{M}_{2n}$ a centro-projective connection, the transport of a puncture in which along a curve L from a point M to a point M_1 is carried out as follows: a puncture u^i at the point M is transformed into the puncture

$$v^i = \frac{F^i_k u^k}{-a_p F^p_j u^j + a_p u^p + 1}, \tag{2}$$

then v^i is transported along the curve L from the point M to the point M_1 in the centro-projective connection $(\Gamma^i_{jk}, \Gamma_{jk})$ and at the point M_1 is transformed into the puncture

$$\tilde{u}^i = \frac{-F^i_j \tilde{v}^j}{a_p F^p_j \tilde{v}^j + a_p \tilde{v}^p + 1}$$

by the transformation inverse to (2).

The object $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ is called the centro-projective connection conjugate to the connection $(\Gamma^i_{jk}, \Gamma_{jk})$ with respect to the copuncture a_i. From (1) it is seen that the subobject $\tilde{\Gamma}^i_{jk}$ of the object of the conjugate centro-projective connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ is an affine connection conjugate to the affine connection Γ^i_{jk} ([2]), which forms the subobject of the original object of centro-projective connection $(\Gamma^i_{jk}, \Gamma_{jk})$.

Theorem 1. *If the centro-projective connection $(\Gamma^i_{jk}, \Gamma_{jk})$ is attached to an affine connection Γ^i_{jk} and a_i is a corresponding copuncture covariantly constant in this connection, then the conjugate centro-projective connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ is also attached to the connection $\tilde{\Gamma}^i_{jk}$ conjugate to the affine connection Γ^i_{jk}.*

Proof. Since $D_j a_k = \partial a_k/\partial x^j - a_p \Gamma^p_{kj} - \Gamma_{kj} = 0$, we have

$$\tilde{\Gamma}_{jk} = \Gamma_{jk} - a_i F^s_j F^i_{s,k} = \Gamma_{jk} - a_i(\tilde{\Gamma}^i_{jk} - \Gamma^i_{jk}) = \frac{\partial a_j}{\partial x^k} - a_p \tilde{\Gamma}^p_{jk}.$$

The connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ is attached to the affine connection $\tilde{\Gamma}^i_{jk}$, and the copuncture a_k is covariantly constant in this connection.

Let us note that, for a covariant tensor of degree one, the analogous property does not hold: a tensor, being covariantly constant in an affine connection—

of Γ^i_{jk}, is not at all obliged to be covariantly constant in the conjugate connection $\tilde{\Gamma}^i_{jk}$.

2. The object of a centro-projective connection

$$\left\{\Gamma^i_{jk}, \quad \Gamma_{jk} = -\frac{1}{2n+1}\left(\frac{\partial \Gamma^m_{mj}}{\partial x^k} - \Gamma^m_{ml}\Gamma^l_{jk}\right)\right\},$$

defined by the copunctor $-\dfrac{1}{2n+1}\Gamma^m_{ml}$, will be called left-invariantly associated with the affine connection Γ^i_{jk}, and the object of a centro-projective connection defined by the copunctor $-\dfrac{1}{2n+1}\Gamma^m_{lm}$, right-invariantly associated with the affine connection Γ^i_{jk}. If the left- and right-invariantly associated objects of a centro-projective connection coincide, then we shall say that the object of a centro-projective connection $(\Gamma^i_{jk}, \Gamma_{jk})$ is invariantly associated with Γ^i_{jk}. It is clear that both objects of a centro-projective connection associated with a symmetric affine connection Γ^i_{jk} coincide.

The connection $\tilde{\Gamma}^i_{jk}$ conjugate to the symmetric affine connection Γ^i_{jk} is, generally speaking, nonsymmetric, and therefore the objects of a centro-projective connection left- and right-invariantly associated with it need not coincide.

Theorem 2. If the connection $(\Gamma^i_{jk}, \Gamma_{jk})$, invariantly associated with a symmetric affine connection Γ^i_{jk}, then the conjugate centro-projective connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ is left-invariantly associated with $\tilde{\Gamma}^i_{jk}$.

Proof. From Theorem 1 it follows that the copunctor $-\dfrac{1}{2n+1}\Gamma^k_{kj}$ is covariantly constant in the connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$, and since $\tilde{\Gamma}^k_{kj} = \Gamma^k_{kj}$, everything is proved.

Consider the tensor $B_j = F^l_s F^s_{j,l}$. A symmetric affine connection Γ^i_{jk} for which $B_j = 0$ is called a normal affine connection. The tensor B_j itself we shall call the normal tensor of the connection Γ^i_{jk}.

Theorem 3. Let $(\Gamma^i_{jk}, \Gamma_{jk})$ be a centro-projective connection invariantly associated with a symmetric affine connection Γ^i_{jk}. In order that the conjugate centro-projective connection $(\tilde{\Gamma}^i_{jk}, \tilde{\Gamma}_{jk})$ be invariantly associated with $\tilde{\Gamma}^i_{jk}$, it is necessary and sufficient that the normal tensor B_j of the connection Γ^i_{jk} be covariantly constant in the conjugate connection $\tilde{\Gamma}^i_{jk}$.

3. The connection

$$\gamma^i_{jk} = \frac{1}{2}(\tilde{\Gamma}^i_{jk} + \tilde{\Gamma}^i_{kj}), \qquad \gamma_{jk} = \frac{1}{2}(\tilde{\Gamma}_{jk} + \tilde{\Gamma}_{kj}) \tag{3}$$

is called the symmetrically conjugate centro-projective connection for the connection $(\Gamma^i_{jk}, \Gamma_{jk})$.

Theorem 4. Let $(\Gamma^i_{jk}, \Gamma_{jk})$ be associated with a symmetric affine connection Γ^i_{jk}, i.e. $\Gamma^i_{kj} = \Gamma^i_{jk}$, and suppose there exists a copunctor a_i such that $\Gamma_{jk} = \partial a_j/\partial x^k - a_p\Gamma^p_{jk}$. In order that the complete curvature object of the connection $(\Gamma^i_{jk}, \Gamma_{jk})$ be equal to zero, it is necessary and sufficient that the symmetrically conjugate centro-projective connection $(\gamma^i_{jk}, \gamma_{jk})$ be associated with the affine connection γ^i_{jk} by means of the copunctor a_i.

Proof. The necessity is obvious.

Sufficiency. Let $\gamma_{jk} = \partial a_j/\partial x^k - a_p\gamma^p_{jk}$. Taking into account relation (3), we obtain

$$\frac{1}{2}\tilde{\Gamma}_{jk} + \frac{1}{2}\tilde{\Gamma}_{kj} = \frac{\partial a_j}{\partial x^k} - \frac{1}{2}a_p\tilde{\Gamma}^p_{jk} - \frac{1}{2}a_p\tilde{\Gamma}^p_{kj},$$

or $\partial a_k/\partial x^j = \partial a_j/\partial x^k$.

Theorem 5. Let a_j be a concircular and $(\Gamma^i_{jk} = \Gamma^i_{kj}, \Gamma^i_{jk})$ a centro-projective connection in which this concircular is covariantly constant. If $(\gamma^i_{jk}, \gamma^i_{jk})$ is the symmetrically conjugate connection for $(\Gamma^i_{jk}, \Gamma^i_{jk})$, and $(\tilde{\gamma}^i_{jk}, \tilde{\gamma}^i_{jk})$ is the conjugate centro-projective connection for the connection $(\gamma^i_{jk}, \gamma^i_{jk})$, then the complete torsion object $(\widetilde{S}^i_{jk}, \widetilde{S}_{jk})$ of the connection $(\tilde{\gamma}^i_{jk}, \tilde{\gamma}_{jk})$ does not depend on the choice of the original connection $(\Gamma^i_{jk}, \Gamma_{jk})$ and is equal to

$$\widetilde{S}^i_{jk} = 2t^i_{jk};$$

$$\widetilde{S}_{jk} = 2a_i t^i_{kj} + \frac{1}{2}\left(\delta^s_j - F^s_j\right)\left(\frac{\partial a_s}{\partial x^k} - \frac{\partial a_k}{\partial x^s}\right) - \frac{1}{2}\left(\delta^s_k - F^s_k\right)\left(\frac{\partial a_s}{\partial x^j} - \frac{\partial a_j}{\partial x^s}\right),$$

where t^i_{jk} is the torsion tensor of the almost complex structure.

4. A normal centro-projective connection is defined as a symmetric centro-projective connection invariantly attached to a normal affine connection.

Theorem 6. If Γ^i_{jk} is an equiaffine connection and γ^i_{jk} is an affine connection symmetrically conjugate to it, then the centro-projective connection invariantly attached to γ^i_{jk} will be normal if and only if the normal tensor B_m of the equiaffine connection Γ^i_{jk} is a gradient.

Proof. Necessity. $\gamma^m_{mj} = \Gamma^m_{mj} - {}^1/_2 B_j$ and $\partial\gamma^m_{mj}/\partial x^k = \partial\gamma^m_{mk}/\partial x^j$; moreover, $\partial\Gamma^m_{mj}/\partial x^k = \partial\Gamma^m_{mk}/\partial x^j$, and therefore $\partial B_j/\partial x^k = \partial B_k/\partial x^j$ and $B_j = \partial\varphi/\partial x^j$, where φ is an arbitrary scalar function.

Sufficiency. a) We shall prove that the normal tensor $\widetilde{B}_l$ of the connection γ^i_{jk} is equal to zero.

$$\widetilde{B}_l = F^k_m\left(\frac{\partial F^m_l}{\partial x^k} + F^p_l\gamma^m_{pk} - F^m_p\gamma^p_{lk}\right) =$$

$$= \frac{1}{2}F^k_m\left(\frac{\partial F^m_l}{\partial x^k} + F^p_l\widetilde{\Gamma}^m_{pk} - F^m_p\widetilde{\Gamma}^p_{lk}\right) + \frac{1}{2}F^k_m\left(\frac{\partial F^m_l}{\partial x^k} + F^p_l\widetilde{\Gamma}^m_{kp} - F^m_p\widetilde{\Gamma}^p_{kl}\right) =$$

$$= \frac{1}{2}B_l + \frac{1}{2}F^k_m F^p_l F^s_p F^m_{s,k} - \frac{1}{2}F^k_m F^m_p F^s_l F^p_{s,k} +$$

$$+\frac{1}{2}B_l + \frac{1}{2}F^k_m F^p_l F^s_k F^m_{s,p} - \frac{1}{2}F^k_m F^m_p F^s_k F^p_{s,l} =$$

$$= B_l - \frac{1}{2}B_l - \frac{1}{2}B_l - \frac{1}{2}F^p_l F^s_{s,p} + \frac{1}{2}F^s_k F^k_{s,l} = 0.$$

b) We shall prove that $\gamma_{jk} = \gamma_{kj}$:

$$\frac{\partial\gamma^m_{mk}}{\partial x^j} - \frac{\partial\gamma^m_{mj}}{\partial x^k} = \frac{\partial\Gamma^m_{mk}}{\partial x^j} - \frac{\partial\Gamma^m_{mj}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial B_k}{\partial x^j} - \frac{\partial B_j}{\partial x^k}\right) = 0.$$

Theorem 7. If $(\Gamma^i_{jk}, \Gamma_{jk})$ is a normal centro-projective connection, then the centro-projective connection $(\gamma^i_{jk}, \gamma_{jk})$ symmetrically conjugate to it is also normal.

The proof follows from Theorem 4, taking into account that $\gamma^m_{mk} = \Gamma^m_{mk}$. The author expresses gratitude to V. G. Lemlein for valuable suggestions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.