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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR ELLIPTIC EQUATIONS DEGENERATING ON THE BOUNDARY OF THE DOMAIN

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The first boundary-value problem for a second-order elliptic equation degenerating on the boundary of a domain was first investigated by M. V. Keldysh ([1]). Later a number of papers appeared ([2,3]) and others, devoted to degenerating second-order elliptic equations. Among elliptic equations of higher orders degenerating on the boundary of a domain, equations to which the variational method is applicable have been considered. Fourth-order elliptic equations degenerating on the boundary of a domain, to which the variational method is not applicable, were considered by V. K. Zakharov ([4]).

In the present paper an elliptic equation on part of the boundary of a domain degenerates into a so-called quasi-parabolic equation. It is proved that the formulation of the problem depends on the sign of the coefficient of the highest derivative of odd order with respect to the variable with respect to which the highest even derivative degenerates.

Let a bounded domain $Q \subset R^n$ be situated in the half-space $x_n > 0$ and let part Γ_0 of its boundary Γ adjoin the plane $x_n = 0$. We denote the remaining part of the boundary by Γ_1: $\Gamma_1 \cup \Gamma_0 = \Gamma$, $\overline{Q} = Q + \Gamma$. We assume that the boundary Γ_1 is such that Sobolev' s embedding theorems hold for it ([5]). In the domain Q consider the equation

$$Lu = L_0 u + Au = h(x), \tag{1}$$

where

$$L_0 u \equiv \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(A_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right) + b\, \frac{\partial^3 u}{\partial x_n^3}$$

is the principal part of the operator L;

$$Au = \sum_{i/4+s/3<1} a_{(i,s)}(x)\, \frac{\partial^s}{\partial x_n^s} D^i u$$

is an operator subordinate to the principal part. Here $s \geq 0$, $i \geq 0$ are some integers; $x = (x_1, \dots, x_n)$; $i = i_1 + \dots + i_{n-1}$; $D^i = \partial^i / \partial x_1^{i_1} \cdots \partial x_{n-1}^{i_{n-1}}$; $b = \pm 1$; the coefficients $A_{ij}(x) = A_{ji}(x)$, $a_{(i,s)}(x)$ are sufficiently smooth functions in $\overline{Q}$, and

$$\sum_{i,j=1}^{n-1} A_{ij}(x) \xi_i^2 \xi_j^2 \geq \theta^2 > 0 \quad \text{for all } \xi_i : \sum_{i=1}^{n-1} \xi_i^2 \neq 0; \tag{2}$$

$$c_{i1}^2 x_n^{\alpha_i} \leq A_{in}(x) \leq c_{i2}^2 x_n^{\alpha_i} \quad (i = 1, \dots, n);$$

α_i are some nonnegative numbers.

For equation (1) the following boundary-value problems are posed:

Problem D. Find a solution $u(x)$ of equation (1) vanishing on Γ together with its first derivatives, if: a) $b = -1$, α_n is arbitrary, or b) $b = +1$, $\alpha_n < 1$.

Problem E. Find a solution $u(x)$ of equation (1) vanishing on Γ_1 together with its first derivatives, if $b = +1$, $\alpha_n \geq 1$; on Γ_0 in this case only the function $u(x)$ vanishes.

Let us complete C_0^4—the set of all functions, four times continuously differentiable in Q, vanishing near Γ—in the metric

$$\|u\|_4^2 = \iint_Q \left[\sum_{i,j=1}^{n} A_{ij}(x) \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \left(\frac{\partial u}{\partial x_n} \right)^2 \right] dQ. \tag{3}$$

We denote the resulting space by $\dot{W}_2^2(\alpha)$. The corresponding embedding theorems, analogous to the theorems in ([2],[4]), are proved; in particular, for any function $u(x) \in \dot{W}_2^2(\alpha)$ the estimate

$$\iint_Q \sigma(x) u^2(x) \, dQ \leq c_3 \|u\|_+^2,$$

holds, where c_3 is a constant independent of $u(x)$, and

$$\sigma(x) = \begin{cases} x_n^{\alpha_n - 4} |\ln x_n|^{-1-\varepsilon_0}, & \text{for } \alpha_n \leq 2, \\ x_n^{-2} |\ln x_n|^{-1-\varepsilon_0}, & \text{for } \alpha_n \geq 2, \end{cases} \qquad (\varepsilon_0 > 0).$$

To prove the existence of the required solutions, known theorems from functional analysis are used (see, for example, ([6])). The method of ([6]) for proving the existence of a generalized solution of degenerating elliptic equations of the second order was used in ([7]). Denote by $\mathcal{L}_2(\sigma^{-1})$ the set of functions $f(x)$ for which

$$\iint_Q \sigma^{-1} f^2(x) \, dQ < +\infty,$$

and by $W_2^{-2}(\alpha)$ the closure of the set $\mathcal{L}_2(\sigma^{-1})$ with respect to the norm

$$\|f\|_- = \sup_{u \in \dot{W}_2^2(\alpha)} \frac{|(f,u)|}{\|u\|_+},$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{L}_2(Q)$.

Lemma 1. The space $W_2^{-2}(\alpha)$ is isometrically isomorphic to the space of all linear bounded functionals on the Hilbert space $\dot{W}_2^2(\alpha)$.

Theorem 1. Every linear bounded functional $m_u(f)$ on the space $W_2^{-2}(\alpha)$ can be represented as

$$m_u(f) = (u, f), \quad u \in \dot{W}_2^2(\alpha).$$

The proofs of Lemma 1 and Theorem 1 are carried out analogously to ([6]).

Denote by $\dot{W}_2^4(Q)$ the set of functions $v(x) \in W_2^4(Q)$ that vanish on the boundary Γ together with their first derivatives, and by $\overset{\vdots}{W}{}_2^4(Q)$ the set of functions $v(x) \in \dot{W}_2^4(Q)$ satisfying on the boundary the following conditions:

$$v|_\Gamma = 0, \qquad \left.\frac{\partial v}{\partial x_i}\right|_{\Gamma_1} = 0 \quad (i = 1, \dots, n).$$

Definition 1. A **weak solution of problem D** for equation (1), for $\alpha_n \geq 1$, will mean a function $u(x) \in \dot{W}_2^2(\alpha)$ that satisfies the equality

$$(h, v) = (u, L^* v) \tag{4}$$

for all functions $v(x) \in \overset{\vdots}{W}{}_2^4(Q)$, where L^* is the "formally adjoint" operator to L.

Weak solutions of problem D for $\alpha_n < 1$, and of problem E with $v(x) \in \dot{W}_2^4(Q)$, are defined in an analogous way.

Theorem 2. If the coefficients of equation (1) for $b = -1$ are such that

$$(L^* v, v) \geq \text{const}\, \|v\|_+^2 \tag{5}$$

for any $v(x) \in \overset{\vdots}{W}{}_2^4(Q)$, then for every function $h(x) \in W_2^{-2}(\alpha)$ there exists a weak solution of problem D for $\alpha_n \geq 1$.

To prove this theorem, the generalized Schwarz inequality is applied to (5), and then the Hahn–Banach theorem and the theorema 1. The existence of a weak solution of problem D for $\alpha_n < 1$ and of problem E is proved in the same way. We note that $\partial u/\partial x_n$ in the case of problem D ($\alpha_n \geqslant 1$) vanishes on Γ_0 in the following sense:

$$\lim_{\delta\to 0}\int_{\Gamma_\delta}\frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}\,d\Gamma_\delta = 0 \qquad (\Gamma_\delta = Q\cap(x_n=\delta))$$

for any function $v(x)\in \dot{W}_2^4(Q)$ that vanishes near Γ_1. The remaining boundary values are assumed in the mean.

Next suppose that $\alpha_n \geqslant 3$ and, for simplicity, $A_{in}(x)=0$ for $i=1,\dots,n-1$. We shall prove the uniqueness of the weak solution of problem D in the cylinder Q, whose lateral surface, upper and lower bases we denote respectively by S,Ω_T,Ω_0, and $\Gamma = S\cup\Omega_T\cup\Omega_0$. Choose $v(x)$ as the solution of the following boundary-value problem:

$$\sum_{i,j=1}^{n-1}\frac{\partial^2}{\partial x_i\partial x_j}\left(A_{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}\right)+\frac{\partial^3 v}{\partial x_n^3}+\frac{1}{2}a_{(0,0)}v = u(x); \qquad (6)$$

$$v\big|_\Gamma = 0, \qquad \frac{\partial v}{\partial x_i}\bigg|_S = 0 \quad (i=1,\dots,n-1), \qquad \frac{\partial v}{\partial x_n}\bigg|_{\Omega_T} = 0. \qquad (7)$$

A solution $v(x)\in\dot{W}_2^4(Q)$ of problem (6)–(7) exists and is unique for $a_{(0,0)}>0$ ([8,9]), and from (4) for $h=0$ we obtain:

$$\iint_Q u^2\,dQ + \iiint_Q\left[\sum_{i,j=1}^{n-1}\frac{\partial^2}{\partial x_i\partial x_j}\left(A_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\right)+\frac{\partial^3 v}{\partial x_n^3}+\frac{1}{2}a_{(0,0)}v\right]\times$$
$$\times\frac{\partial^2}{\partial x_n^2}\left(A_{nn}(x)\frac{\partial^2 v}{\partial x_n^2}\right)dQ$$
$$+\iiint_Q\left[\sum_{i,j=1}^{n-1}\frac{\partial^2}{\partial x_i\partial x_j}\left(A_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}\right)+\frac{\partial^3 v}{\partial x_n^3}+\frac{1}{2}a_{(0,0)}v\right]\left[A^*v-\frac{1}{2}a_{(0,0)}v\right]dQ = 0,$$

where A^* is the "formally adjoint" operator to A.

Transforming the integrals by integration by parts, estimating them from below by known inequalities, and applying Kudryavtsev' s lemma ([2]), we find:

$$\left(1-\sum_{i=1}^{N}\varepsilon_i\right)\iint_Q u^2\,dQ+\iint_Q\left[\frac{1}{4}a_{(0,0)}^2-\sum_{i=1}^{N}c(\varepsilon_i)\right]v^2\,dQ\leqslant 0, \qquad (8)$$

where the ε_i are chosen so that $1-\sum_{i=1}^N\varepsilon_i>0$, and N is a certain integer.

Thus, from (8) we obtain the following uniqueness theorem:

Theorem 3. If

$$a^2_{(0,0)} \geqslant 4 \sum_{i=1}^{N} c(\varepsilon_i) \tag{9}$$

then the weak solution of problem D is unique.

Consider equation (1) for $b = +1$ in the cylinder Q. The method of proving the uniqueness theorem used for $b = -1$ does not essentially work for $b = +1$, since the corresponding boundary-value problem for $v(x)$ is not solvable. The initial domain of definition of the operator of the left-hand side consists of functions in C^4 satisfying the conditions:

$$u\big|_\Gamma = 0, \qquad \frac{\partial u}{\partial x_i}\bigg|_S = 0 \quad (i = 1, \dots, n-1), \qquad \frac{\partial u}{\partial x_n}\bigg|_{\Omega_T} = 0. \tag{10}$$

The closure of the set of smooth functions satisfying conditions (10) in the norm (3) gives $\dot{W}^2_2(\alpha)$. We extend the operator L, defining it on all $u(x) \in \dot{W}^2_2(\alpha)$ satisfying the additional condition

$$\frac{\partial^2 u}{\partial x_n^2} \in \mathcal{L}_2(Q) \tag{11}$$

as a functional on $\dot{W}^2_2(\alpha)$, by setting

$$\langle Lu, v\rangle = -\iint_Q \frac{\partial^2 u}{\partial x_n^2}\frac{\partial v}{\partial x_n}\, dQ + \iint_Q \sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\, dQ + A(u,v) \tag{12}$$

for any $v(x) \in \dot{W}^2_2(\alpha)$, where $A(u,v)$ is the expression obtained formally from $\iint_Q Au \cdot v\, dQ$ by integration by parts. The extension constructed is closed in $W^{-2}_2(\alpha)$.

Definition 2. We shall say that $u(x) \in B^C_L$ (the domain of definition of the strong extension of the operator L) if $u \in \dot{W}^2_2(\alpha)$ and there exist an element $h(x) \in W^{-2}_2(\alpha)$ and a sequence u_i of functions belonging to $\dot{W}^2_2(\alpha)$ and satisfying the additional condition (11), such that $\|u - u_i\|_+ \to 0$, $\|Lu_i - h\|_- \to 0$ as $i \to \infty$, where L is understood in the sense of (12). The corresponding function will be called a **strong solution of problem E**.

Theorem 4. *The strong solution of problem E, for $a_{0,0}$ satisfying condition (9), exists and is unique.*

Uniqueness follows from the following lemma:

Lemma 2. *For strong solutions of problem E the inequality*

$$\|u\|_+ \leq \text{const}\, \|Lu\|_-$$

holds.

The existence of a strong solution of problem E is proved in the same way as in ([10]), using Theorem 3.

Remark 1. The uniqueness of the generalized solution of problem D for $\alpha_n < 1$ was proved by another method in ([4]).

Remark 2. The results obtained also extend to the corresponding elliptic equations of higher orders that degenerate on the boundary of the domain; we have considered fourth-order equations only for simplicity of exposition.

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Note: Figure translations are in progress. See original paper for figures.