I. S. Iokhvidov

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Abstract

Full Text

I. S. Iokhvidov

On a Lemma of K. Fan Generalizing A. N. Tikhonov' s Fixed-Point Principle

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1. In K. Fan' s paper (1), along with other interesting generalizations and modifications of the fixed-point principle, the following very simply proved lemma is contained.

Lemma (K. Fan). *Let $\mathfrak{X}$ be a nonempty bicompact set in a separable topological vector space, and let $\mathfrak{A}$ be a closed subset of $\mathfrak{X} \times \mathfrak{X}$ possessing the following two properties: 1) for every $x \in \mathfrak{X}$ the set $\mathfrak{A}$ contains the pair (x, x); 2) for any $x \in \mathfrak{X}$ the set $\{y \in \mathfrak{X} : (x, y) \notin \mathfrak{A}\}$ is convex (or empty). Then there exists a point $x_0 \in \mathfrak{X}$ such that $(x_0, y) \in \mathfrak{A}$ for all $y \in \mathfrak{X}$.*

In the same paper (1) it is shown how, from this lemma, in the case of a locally convex topological space (2), A. N. Tikhonov' s fixed-point principle (3) follows immediately. An analysis of this argument of K. Fan makes it possible to conclude that, in an analogous way, for locally convex topological spaces one can obtain a theorem somewhat more general than A. N. Tikhonov' s principle. This theorem 1 is given below, some of its applications are indicated, as well as those questions where it may find further applications.

2. **Theorem 1.** *Let there be given a locally convex separable topological space $\mathfrak{E}$ and in it some convex bicompact set $\mathfrak{X}$. Let $z = F(x, y)$ be a continuous (jointly in the variables x, y) mapping $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{E}$, affine in y in the sense that for all $x, y_1, y_2 \in \mathfrak{X}$ and $\alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1$,*

$$F(x, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 F(x, y_1) + \alpha_2 F(x, y_2), \tag{1}$$

and possessing the property that for every $x \in \mathfrak{X}$ the equation $F(x, y) = 0$ has at least one solution $y \in \mathfrak{X}$.

Then there exists an element $x_0 \in \mathfrak{X}$ such that $F(x_0, x_0) = 0$.

It is easy to see that, for $F(x, y) = y - \varphi(x)$, where $\varphi(x)$ is a continuous mapping $\mathfrak{X} \to \mathfrak{X}$, theorem 1 becomes A. N. Tikhonov' s principle.

Proof of Theorem 1. Following K. Fan' s idea, consider the totality of all continuous seminorms (2) in $\mathfrak{E}$: $\{p_\nu\}_{\nu \in I}$, and for fixed $\nu \in I$ denote by $\mathfrak{Z}_\nu$ the totality of those $x \in \mathfrak{X}$ for which $p_\nu(F(x, x)) = 0$. Our aim, obviously, is to find a point

$$x_0 \in \bigcap_{\nu \in I} \mathfrak{Z}_\nu.$$

Since, by virtue of the continuity of the mapping $F(x, y)$ and of the seminorms p_ν $(\nu \in I)$, all the sets $\mathfrak{Z}_\nu$ are closed, and $\mathfrak{X}$ is bicompact, it is enough to prove that the family $\{\mathfrak{Z}_\nu\}_{\nu \in I}$ is centered, i.e., for any finite set $\nu_1, \nu_2, \dots, \nu_n \in I$ we have

$$\bigcap_{k=1}^{n} \mathfrak{Z}_{\nu_k} \neq \emptyset.$$

Fix a set of indices $\nu_1, \nu_2, \dots, \nu_n \in I$ and denote by $\mathfrak{A}$ the totality of all pairs $(x, y) \in \mathfrak{X} \times \mathfrak{X}$ for which

$$\sum_{k=1}^{n} p_{\nu_k}(F(x, y)) \geq \sum_{k=1}^{n} p_{\nu_k}(F(x, x)). \tag{2}$$

The set $\mathfrak{A}$ is obviously closed in $\mathfrak{E} \times \mathfrak{E}$ and contains all pairs of the form (x, x) $(x \in \mathfrak{X})$. Further, if for some fixed $x \in \mathfrak{X}$ and some $y_1, y_2 \in \mathfrak{X}$ we have $(x, y_1) \notin \mathfrak{A}$ and $(x, y_2) \notin \mathfrak{A}$, i.e.

$$\sum_{k=1}^{n} p_{\nu_k}(F(x, y_i)) < \sum_{k=1}^{n} p_{\nu_k}(F(x, x)) \qquad (i = 1, 2),$$

then from property (1) of the function $F(x, y)$ and the triangle inequality for seminorms it immediately follows that, for any $\alpha_1, \alpha_2 \geq 0$ $(\alpha_1 + \alpha_2 = 1)$, we have

$$\sum_{k=1}^{n} p_{\nu_k}(F(x, \alpha_1 y_1 + \alpha_2 y_2)) < \sum_{k=1}^{n} p_{\nu_k}(F(x, x)),$$

i.e. $(x, \alpha_1 y_1 + \alpha_2 y_2) \notin \mathfrak{A}$, so that all the conditions of K. Fan' s lemma are satisfied. By this lemma there exists an element $x_0 \in \mathfrak{X}$ such that $(x_0, y) \in \mathfrak{A}$ for all $y \in \mathfrak{X}$. In particular, taking as y a solution $y = y_0 \in \mathfrak{X}$ of the equation $F(x_0, y) = 0$, we have $(x_0, y_0) \in \mathfrak{A}$, where $F(x_0, y_0) = 0$. Putting in (2) $x = x_0$, $y = y_0$, we have $p_{\nu_k}(F(x_0, x_0)) = 0$ $(k = 1, 2, \dots, n)$ and

$$x_0 \in \bigcap_{k=1}^{n} \mathfrak{Z}_{\nu_k} \neq \emptyset.$$

3. Theorem 1 finds an application, for example, in the case where $\mathfrak{E}$ is the space $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ of all continuous linear mappings of a Banach space $\mathfrak{B}_1$ into a Banach space $\mathfrak{B}_2$. It is known that, endowed with the weak operator topology ([4]), the space $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is a locally convex separated topological space. Applying Theorem 1 to it, we immediately obtain

Theorem 2. *Let in the space $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ there be given a convex bicompact set $\mathfrak{X}$, on which two mappings $\varphi(X)$ and $\psi(X) : \mathfrak{X} \to \mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ are defined, possessing the following properties: 1) for every $X \in \mathfrak{X}$ there exists an operator $[\varphi(X)]^{-1} \in \mathfrak{L}(\mathfrak{B}_2, \mathfrak{B}_1)$ such that $Y = \psi(X) \cdot [\varphi(X)]^{-1} \in \mathfrak{X}$; 2) the mapping $F(X, Y) = Y \cdot \varphi(X) - \psi(X)$ of the set $\mathfrak{X} \times \mathfrak{X}$ into $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ is continuous (in the weak operator topology). Then there exists an operator $X_0 \in \mathfrak{X}$ such that $X_0 = \psi(X_0) \cdot [\varphi(X_0)]^{-1}$.*

Corollary 1. *Suppose that: 1) the convex bicompact set $\mathfrak{X}$ $(\in \mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2))$ from Theorem 2 is bounded in the uniform operator topology ([4]); 2) the function $\psi(X)$, mapping $\mathfrak{X}$ into $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$, is continuous in the weak operator topology; 3) the function $\varphi(X)$ is a "polynomial" of degree n of the form*

$$\varphi(X) = \sum_{j=1}^{m} (A_0^{(j)} + B_1^{(j)} X A_1^{(j)} + B_2^{(j)} X B_3^{(j)} X A_2^{(j)} + \dots + B_p^{(j)} X B_{p+1}^{(j)} X \dots B_{p+n}^{(j)} X A_n^{(j)}), \tag{3}$$

where $A_i^{(j)}, B_k^{(j)} \in \mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$ $(j = 1, 2, \dots, m;\ i = 1, 2, \dots, n;\ k = 1, 2, \dots, p + n;\ p = [n(n-1) + 2]/2)$, and the operators $B_1^{(j)}, B_2^{(j)}, \dots, B_p^{(j)}, \dots, B_{p+n}^{(j)}$ are "completely continuous" $(j = 1, 2, \dots, m)$. If for every $X \in \mathfrak{X}$ there exists an operator $[\varphi(X)]^{-1} \in \mathfrak{L}(\mathfrak{B}_2, \mathfrak{B}_1)$ such that $Y = \psi(X) \cdot [\varphi(X)]^{-1} \in \mathfrak{X}$, then there exists an operator $X_0 \in \mathfrak{X}$ for which $X_0 = \psi(X_0) \cdot [\varphi(X_0)]^{-1}$.

Indeed (cf. ([5])), approximating in the uniform operator topology the "completely continuous" operators $B_k^{(j)}$ from (3) by finite-dimensional oper-

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* By "completely continuous"we mean here an operator admitting approximation in the uniform operator topology by finite-dimensional operators. In the case of Banach spaces with a basis this definition is equivalent to the usual definition of a completely continuous operator.

operators, it is easy to see that we obtain an approximation of the function $Y \cdot \varphi(X)$ by functions $Y \cdot \varphi_\nu(X)$ continuous in the weak operator topology; moreover, by condition 1), this approximation will be uniform with respect to $X, Y \in \mathfrak{X}$, so that the function $Y \cdot \varphi(X)$, and hence also (by 2)) $F(X, Y) = Y \cdot \varphi(X) - \psi(X)$, are continuous in the weak operator topology in the aggregate of the variables X, Y. It remains to apply Theorem 2.

As the function $\psi(X)$ in Corollary 1 one may take, for example, operator "polynomials" of any degree of the same form (3) as $\varphi(X)$, while discarding the requirement of "complete continuity" of the first "operator coefficients" $B_1^{(j)}, B_2^{(j)}, B_4^{(j)}, \dots, B_p^{(j)}$ in each of the summands standing in parentheses under the summation sign in (3). It is clear that condition 2) for $\psi(X)$ will then be satisfied. Thus $Y = \psi(X) \cdot [\varphi(X)]^{-1}$ is in this case a "rational" function of a special type. This opens prospects for investigating, by means of Theorem 2, questions of the existence of operator "roots" of certain operator "polynomials."

Corollary 2. *If in Corollary 1 one additionally requires that the Banach space $\mathfrak{B}_2$ be reflexive, and condition 1) is replaced by the requirement that the convex set $\mathfrak{X}$ be bounded and closed in the uniform operator topology, then the assertion of Corollary 1 remains valid.*

Indeed, it is known (([4], p. 551)) that the closed unit ball in $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_2)$, in the case when the space $\mathfrak{B}_2$ is reflexive, is bicompact in the weak operator topology.

Let us note that neither in Theorem 2 nor in its corollaries could one, without additional investigation, use Tikhonov' s principle instead of Theorem 1, since it is not clear whether, under our conditions, the function $Y = \psi(X) \cdot [\varphi(X)]^{-1}$ will be continuous.

4. In a recently published note ([6]) K. Fan indicated how the application of his lemma makes it possible to obtain theorems on the existence of invariant subspaces of certain operators in linear topological spaces related to the spaces Π_χ ([7]). As special cases of these theorems one obtains the known results ([7–12]) on invariant subspaces of certain classes of operators in spaces with an indefinite metric of finite rank of indefiniteness (Π_χ and some others *). However, K. Fan' s theorems from ([6]) apply only to operators in $\mathfrak{L}(\mathfrak{B}_1, \mathfrak{B}_1 \times \mathfrak{B}_2)$, where $\dim \mathfrak{B}_1 < \infty$, and this restriction is essential in his arguments. Therefore the strongest of the results obtained up to the present time ([13,14,5]) on invariant subspaces of operators acting in spaces more general than Π_χ and related to it cannot be derived from them. At the same time, as Theorems 1 and 2 show, K. Fan' s lemma can be of service also in these questions, to which we now turn.

At the end of his paper ([5]) M. G. Krein briefly indicates that his Theorem 1 can be reformulated and proved for Banach spaces with an indefinite metric. The results of the present note make it possible to do this with a certain economy of means and at the same time, in one respect, to supplement the formulation somewhat even for the case of a Hilbert space ([5]) or of the Banach space considered in ([12]).

Let $\mathfrak{B}$ be a Banach space, and let $\mathfrak{B} = \mathfrak{B}_1 \dot{+} \mathfrak{B}_2$ be its decomposition into a direct sum of subspaces $\mathfrak{B}_i = P_i\mathfrak{B}$, where P_i $(i = 1, 2)$ are the projectors corresponding to this decomposition. Introduce in the space $\mathfrak{B}$ the so-called J-metric, i.e. the quadratic functional $J(x, x) = \|P_1x\|^2 - \|P_2x\|^2$, with the aid of which we define in $\mathfrak{B}$ J-nonnegative

* Let us note that K. Fan apparently was not aware of the papers ([11,12]) and, in particular, of the fact that not all results of M. L. Brodskii' s theorem ([12])

follow from his theorems.

and, in particular, maximal J-nonnegative subspaces (cf. ([5])). Denote by $\mathfrak{M}_+$ the set of all maximal nonnegative subspaces of $\mathfrak{B}$.

Theorem 3. *Let, in a reflexive Banach space* $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$ $(\mathfrak{B}_i = P_i\mathfrak{B},\ i = 1, 2)$ *with a J-metric, there be given a bounded linear operator A possessing the property that* $J(Ax, Ax) \geq 0$, *whenever* $J(x, x) \geq 0$ $(x \in \mathfrak{B})$. *If, moreover, the operator P_1AP_2 is "completely continuous," then in $\mathfrak{B}$ there exists a subspace* $\Omega_+ \in \mathfrak{M}_+$ *such that* $A\Omega_+ \subset \Omega_+$. *If, in addition, it is known that the operator A maps at least one subspace* $\Omega_0 \in \mathfrak{M}_+$ *into* $A\Omega_0 \in \mathfrak{M}_+$ *and* $Ax \neq 0$ *for* $x \neq 0$ $(J(x, x) \geq 0)$, *then there exists* $\Omega_+ \in \mathfrak{M}_+$ *such that* $A\Omega_+ = \Omega_+$.

What is new here, in comparison with ([5],[12]), is the first assertion of Theorem 3, whose proof is obtained by replacing, in the arguments of M. G. Krein ([5]), Tikhonov' s principle by Theorem 1 of the present note; the second assertion repeats the formulation of Theorem 1 of M. G. Krein ([5]) and is proved by a simple comparison of the first assertion with proposition a) from ([5]), which remains valid also in a Banach space.

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Odessa Civil Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.