A. V. KUZHEL

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Abstract

Full Text

A. V. KUZHEL

ON NONSELFADJOINT OPERATORS GENERATED BY JACOBI MATRICES

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In this note an equation is found for the nonreal eigenvalues of the nonselfadjoint operator A_θ generated by a Jacobi matrix; a completeness condition is established for the system of eigenvectors of the operator A_θ, and a condition is also obtained for the unitary equivalence of certain (simple) parts of nonselfadjoint operators generated by different Jacobi matrices.

1. Let A' be the linear operator defined in the separable Hilbert space H by means of the Jacobi matrix

$$\left\| \begin{matrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{matrix} \right\| \qquad (b_k > 0,\ \bar{a}_k = a_k,\ k = 0, 1, 2, ...) \tag{1}$$

by the relation

$$A'e_k = b_{k-1}e_{k-1} + a_{ke}k + b_{ke_{k+1}} \quad (k = 0, 1, 2, \ldots;\ b_{-1} = 0),$$

where $\{e_k\}_{k=0}^{\infty}$ is an orthonormal basis in H. As is known, the operator A'^* adjoint to A' (which in what follows we shall denote by A) exists and is either selfadjoint, or a symmetric operator with defect index $(1, 1)$. In what follows the latter case is considered.

Then (see, for example, (^{1}), p. 639, or (^{2}), p. 175) the polynomials $P_k(\lambda)$ ($k = 0, 1, 2, ...$), defined by the relations

$$b_{k-1}P_{k-1}(\lambda) + a_{kP}k(\lambda) + b_{kP_{k+1}}(\lambda) = \lambda P_k(\lambda)$$

$$\left(k = 1, 2, 3, \ldots; \quad P_0(\lambda) = 1,\ P_1(\lambda) = \frac{\lambda - a_0}{b_0}\right),$$

satisfy the condition

$$\sum_{k=0}^{\infty} |P_k(\lambda)|^2 < \infty \qquad (\operatorname{Im}\lambda \neq 0).$$

Here the defect subspace $\mathfrak{N}_\lambda$ of the operator A is spanned by the vector

$$g_\lambda = \sum_{k=0}^{\infty} P_k(\bar{\lambda}) e_k.$$

Let us note that the polynomial $P_k(\lambda)$ is related to the characteristic polynomial of the "truncated" Jacobi matrix

$$J_k = \left\| \begin{array}{ccccc} a_0 & b_0 & 0 & \cdots & 0 \\ b_0 & a_1 & b_1 & \cdots & 0 \\ 0 & b_1 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & a_{k-1} \end{array} \right\|$$

by the relation

$$P_k(\lambda) = \frac{\det(\lambda E - J_k)}{b_0 b_1 \dots b_{k-1}},$$

where E is the identity matrix of order k.

In what follows an important role will be played by the function

$$h(\lambda, \mu) = \sum_{k=0}^{\infty} P_k(\lambda) P_k(\mu),$$

which we shall call the **polynomial kernel** of the system $\{P_k(\lambda)\}_{k=0}^{\infty}$.

2. Let D_A be the domain of definition of the operator A $(= A^*)$. Consider the linear manifold D_{A_θ}, which, in addition to the manifold D_A, also contains the vector $\psi = \theta g_i + g_{-i}$, where θ is an arbitrary fixed complex number. Define on D_{A_θ} the operator A_θ by the relation

$$A_\theta f = A^* f \qquad (f \in D_{A_\theta}).$$

Then

$$A_\theta^* = A_{1/\bar{\theta}},$$

whence, in particular, it follows that the operator A_θ is self-adjoint if and only if $|\theta| = 1$.

In what follows we assume that $|\theta| \neq 1$. Then the operator A_θ is a K^1_Π-operator (3), and, consequently, the results of (3,4) may be applied to it. In particular, the α-basis of the operator A_θ consists of the vector $g = \tau g_{-i}$, where

$$\tau = \left|1 - |\theta|^2\right|^{1/2} \left(2 \sum_{k=0}^{\infty} |P_k(i)|^2 \right)^{-1/2},$$

and the corresponding coefficient J is determined by the relation $J = \operatorname{sign}(1 - |\theta|^2)$. This makes it possible to compute the characteristic function $\chi_{A_\theta}(\lambda)$ of the operator A_θ, which in our case is determined by the relation (3)

$$\chi_{A_\theta}(\lambda)\chi_{A_\theta}(i) = 1 + i(\lambda + i)((A^*_\theta - iI)(A^*_\theta - \lambda I)^{-1} g, g)J.$$

As a result we obtain that

$$\chi_{A_\theta}(\lambda) = \frac{\omega(\lambda, \theta)}{\theta\, \omega(\lambda, 1/\bar{\theta})},$$

where

$$\omega(\lambda, t) = (\lambda + i)\, t\, h(\lambda, -i) + (\lambda - i)\, h(\lambda, i)$$

($h(\lambda, \mu)$ is the polynomial kernel of the system $\{P_k(\lambda)\}_{k=0}^{\infty}$).

Using now the results of (3), we obtain the following assertion:

Theorem 1. The non-real spectrum of the operator A_θ coincides with the set of non-real zeros of the function $\omega(\lambda, \theta)$. Moreover, the multiplicity of an arbitrary non-real eigenvalue of the operator A_θ is equal to 1.

3. Let $|\theta| < 1$. Then the operator A_θ is dissipative (i.e., for every f from D_{A_θ}, $\operatorname{Im}(A_\theta f, f) \geq 0$). If $|\theta| > 1$, instead of the operator A_θ one may consider the operator $A^*_\theta = A_{1/\bar{\theta}}$, which, by the preceding, will be dissipative. Consequently, without loss of generality, we may consider only the case $|\theta| < 1$.

Theorem 2. The non-real spectrum $\{\lambda_k\}_{k=1}^{N}$ of the dissipative operator A_θ satisfies the condition

$$\prod_{k=1}^{N} \left| \frac{\bar{\lambda}_k + i}{\lambda_k + i} \right| \geq |\theta| \qquad (N \leq \infty). \tag{2}$$

Moreover, if the operator A_θ is simple (^{3}), then the system of eigenvectors of this operator will be complete in the space H if and only if equality holds in relation (2).

Let us note that in the case when the system of eigenvectors of the operator A_θ is complete in the space H (i.e., when equality holds in (2) and $N = \infty$), the discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$ of the operator A_θ satisfies the condition

$$\sum_{k=1}^{\infty} \operatorname{Im} \lambda_k = \infty.$$

In this case the set $\{\lambda_k\}_{k=1}^{\infty}$ is unbounded.

In the same case, when $\{\lambda_k\}_{k=1}^{\infty}$ is a bounded set, the series

$$\sum_{k=1}^{\infty} \operatorname{Im} \lambda_k$$

converges. Consequently, in this case, on the basis of the preceding, the system of eigenvectors of the operator A_θ cannot be complete in H.

4. Let us consider, in addition to the operator A_θ, the operator $\widetilde{A}_\theta$, which is generated by the matrix

$$\left\| \begin{matrix} \tilde{a}_0 & \tilde{b}_0 & 0 & 0 & \ldots \\ \tilde{b}_0 & \tilde{a}_1 & \tilde{b}_1 & 0 & \ldots \\ 0 & \tilde{b}_1 & \tilde{a}_2 & \tilde{b}_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\| \qquad (\tilde{b}_k > 0,\ \tilde{a}_k = a_k,\ k = 0, 1, 2, \ldots),$$

analogously to how the operator A_θ is generated by the matrix (1). Let, further,

$$\tilde{h}(\lambda, \mu) = \sum_{k=0}^{\infty} \widetilde{P}_k(\lambda) \widetilde{P}_k(\mu)$$

be the polynomial kernel of the system $\{\widetilde{P}_k(\lambda)\}_{k=0}^{\infty}$. Then, using the preceding results and the results of paper (^{3}), we obtain the following assertion:

Theorem 3. Let the polynomial kernels $h(\lambda, i)$ and $\tilde{h}(\lambda, i)$ of the systems $\{P_k(\lambda)\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k(\lambda)\}_{k=0}^{\infty}$ coincide. Then the simple parts of the operators A_θ and $\widetilde{A}_\theta$ are isomorphic.

Thus, by virtue of the preceding theorem, the simple part of the operator A_θ is determined by the polynomial kernel up to isomorphism. Moreover, the simple part of the operator A_θ can be constructively recovered (up to isomorphism) if the polynomial kernel is given.

Uman State
Pedagogical Institute

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Note: Figure translations are in progress. See original paper for figures.