B. Yu. Sternin

1964

SovietRxiv

Abstract

Full Text

B. Yu. Sternin

General Boundary-Value Problems for Elliptic Equations in a Domain Whose Boundary Consists of Manifolds of Different Dimensions

(Presented by Academician I. G. Petrovskii, June 15, 1964)

1. Boundary-value problems in a domain bounded by manifolds of different dimensions were considered by S. L. Sobolev (^{7}). He proved, by the variational method, the unique solvability of the Dirichlet problem for the polyharmonic equation $\Delta^m u = 0$ in the class of functions $u \in W_2^{(m)}$. The functions prescribed on the boundary of the domain were assumed by him to be admissible, i.e., such that there exists at least one $W_2^{(m)}$-extension into the domain. Later L. N. Slobodetskii (^{6}) gave an intrinsic characterization of the set of admissible functions.

2. Let R^N be a Euclidean space of dimension N. Decompose it into the direct sum

$$R^N = R^n \oplus R^\nu \quad (n + \nu = N).$$

We shall denote the elements of the spaces R^ν and R^n, respectively, by $(t^1, \dots, t^\nu)$ and $(x^1, \dots, x_n)$. A point of the space dual (with respect to the Fourier transform) to R^ν and R^n is the sequence of numbers $(\tau_1, \dots, \tau_\nu)$ and $(\xi_1, \dots, \xi_n)$. The duality relations define the bilinear forms

$$(t, \tau) = \sum t^n \tau_k; \qquad (x, \xi) = \sum x^k \xi_k.$$

Let s be an arbitrary real number. By $H^s = H^s(R^N)$ we shall understand the functional space obtained by completing the smooth functions in the norm

$$\|f\|_s = \left(\int (1 + |\xi|^2 + |\tau|^2)^s |f(\xi, \tau)|^2 d\xi \, d\tau \right)^{1/2}.$$

(For brevity, functions of the dual space will be denoted by the same letter as those of the original space, but as functions of the dual argument.)

It is known that H^{-s} is adjoint to H^s. If Ω is a bounded domain, then the space $H^s(\Omega)$ is defined as the space of generalized functions that are restrictions to Ω of functions $f_1 \in H^s(R^N)$. The norm of the space is given by

$$\|f, \Omega\|_s = \inf_{f_1|_\Omega = f} \|f_1\|_s.$$

The space dual to $H^s(\Omega)$ is the space $H^{-s}(\Omega)$, consisting of generalized functions with supports in $\overline{\Omega}$.

3. Consider in Ω the differential expression

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

with smooth complex-valued coefficients; α is a multi-index

$$\alpha = (\alpha_1, \dots, \alpha_N); \qquad |\alpha| = \sum \alpha_k; \qquad x = (x^1, \dots, x^N);$$

$$D^\alpha = (-i)^{|\alpha|} \partial^{\alpha_1} / \partial x_1^{\alpha_1} \dots \partial^{\alpha_N} / \partial x_N^{\alpha_N}.$$

Let the boundary $\partial\Omega$ of the domain Ω consist of smooth closed nonintersecting manifolds without boundary $\partial\Omega^\nu$ of dimension ν:

$$\partial\Omega = \bigcup_{\nu < 2m} \partial\Omega^\nu.$$

Some of the manifolds may be absent, but a manifold of codimension 1 must necessarily be present.

On each of the manifolds $\partial\Omega^\nu$ there is given a system $B^\nu = \{B_j^\nu\}$, $|j| \leq l_\nu$, of singular integro-differential operators $^{(2)}$ with smooth complex-valued coefficients (j is a multi-index, $j = (j_1, \dots, j_\nu)$, $|j| = \sum j_k$). Denote by B the system of all boundary operators, and let $\mathfrak{A} = (A, B)$ be the bounded operator effecting the mapping

$$H^s(\Omega) \to K^s(\Omega, \partial\Omega) = H^{s-2m}(\Omega) \times \prod_{j,\nu} H^{s - m_j^\nu - \nu/2}(\partial\Omega^\nu),$$

$$\max_{j,\nu} \left(m_j^\nu + \frac{\nu}{2} \right) < s < 2m - \frac{\nu}{2}. \tag{1}$$

We now define the number l_ν. $l_1 = m$. For each fixed $\nu > 1$ the number l_ν is a step function of s, equal to $r[2m - s - \nu/2]$, if $2m - s - \nu/2$ is not an integer, and to $2m - s - \nu/2 - 1$ otherwise.

We shall assume that the operator $\mathfrak{A}$ is elliptic. This means:

1) $A_0(x, \lambda) \neq 0$ for $x \in \Omega$, $|\lambda| \neq 0$.

2) A and B_j^1 are connected at each point of the $\partial\Omega^1$-boundary by the Shapiro-Lopatinskii condition $^{(4,8)}$.

3) Let $y \in \partial\Omega^\nu$ ($\nu > 1$). Make an infinitely differentiable change of coordinates after which the equations of the boundary manifold in some neighborhood of the point y have the form $t = 0$ ($t_1 = 0, \dots, t_\nu = 0$). Let $A_0(D_t, D_x)$ and $B_{j0}^\nu(D_t, D_x)$ be the principal parts of the operators A and B_j^ν, whose coefficients are fixed at the point y. We require that the boundary-value problem

$$A_0(D_t, \xi)u = 0, \qquad t \neq 0; \tag{2}$$

$$B^{\nu}_{j0}(D_t, \xi)u = g^{\nu}_j(\xi), \qquad t = 0, \tag{3}$$

be uniquely solvable in $H^s(R^N)$ for $|\xi| \neq 0$ for arbitrary $g^{\nu}_j(\xi)$. This condition admits an adequate formulation in algebraic terms. Indeed, it can be shown that, if $|\xi| \neq 0$, then the space of solutions of equation (2) belonging to H^s is finite-dimensional and has as a basis the fundamental solution and all its derivatives up to order $l_\nu - 1$. In other words, any solution u of equation (2) belonging to H^s has the form

$$u = P(D)\mathcal{E}, \tag{4}$$

where $P(D)$ is a differential operator of order l_ν, and $\mathcal{E}$ is the fundamental solution of equation (2). To determine the coefficients of the operator $P(D)$, one must substitute (4) into the boundary conditions (3). A system of linear algebraic equations is obtained for the coefficients of the operator $P(D)$. The condition for its unique solvability

$$\mathcal{D} = \det \left\| \int \frac{\tau^k B_j(\tau, \xi)}{A_0(\tau, \xi)} \, d\tau \right\| \neq 0 \tag{5}$$

is equivalent to condition 3).

Remark 1. If the coefficients of the operator A are real, then condition 3) is satisfied for the Dirichlet problem: $\partial^j / \partial t^j = \varphi_j(x)$, $|j| \leq m - [\nu/2] - 1$. The determinant $\mathcal{D}$ in this case becomes the determinant

$$\det \left\| \int \tau^i \tau^j A_0^{-1}(\tau, \xi) \, dt \right\|,$$

which is the Gram determinant for the functions $1, \tau, \tau^2, \dots, \tau^{m-[\nu/2]-1}$. Recall that $\tau = (\tau_1, \dots, \tau_\nu)$ and $\tau^\alpha = (\tau_1^{\alpha_1}, \dots, \tau_\nu^{\alpha_\nu})$. These functions are linearly independent, and consequently their Gram determinant is different from zero.

Remark 2. It can be shown that if $s \geq m - \nu/2 + [\nu/2]$, then as the operators B^{ν}_j satisfying condition (5) one may take purely differential operators; whereas if $s < m - \nu/2 + [\nu/2]$, then there is no purely differential boundary-value problem satisfying this condition.

4. A left (right) regularizer of the operator $\mathfrak{A} : H^s \to K^s$ is an operator $\mathfrak{R} : K^s \to H^s$ such that

$$\mathfrak{R}\mathfrak{A} = 1_{K^s} + T_{K^s} \quad (\mathfrak{A}\mathfrak{R} = 1_{H^s} + T_{H^s}),$$

where $1_{K^s}, 1_{H^s}$ are the identity operators, and T_{K^s}, T_{H^s} are completely continuous operators in the spaces K^s and H^s, respectively. The operator $\mathfrak{A}$ is called a Φ-operator if its range is closed and its nullspace and cokernel are finite-dimensional. The main theorem is the following.

Theorem 1. *The following conditions are equivalent:*

1) *The operator $\mathfrak{A}$ is elliptic.*
2) *The operator $\mathfrak{A}$ has left and right regularizers.*
3) *The operator $\mathfrak{A}$ is a Φ-operator from H^s into K^s.*
4) *The following inequality holds*

$$\|u, \Omega\|_s \leq C \left(\|Au, \Omega\|_{s-2m} + \sum_{j,\nu} \|B_j^\nu u, \partial\Omega^\nu\|_{s-m_j^\nu-\nu/2} + \|u, \Omega\|_{s-1} \right).$$

5. It is well known that in the case when the domain is bounded only by a manifold of codimension 1, infinite differentiability of the right-hand sides implies infinite differentiability of the solution. If the boundary of the domain includes manifolds of codimension greater than 1, then the solution may fail to be infinitely differentiable even for infinitely differentiable right-hand sides. We note that one can prove infinite differentiability of the solution in tangential directions.

6. For sufficiently smooth solutions of problem (2), (3) the following holds.

Theorem 2. *Let $u \in H^l$ be a solution of problem (2), (3) for $l \geq 2m - \nu/2$, and let $f \in H^{l+k-2m}$, $g_j^1 \in H^{l+k-m_j^1-1/2}$. Then $u \in H^{l+k}$ and the inequality holds*

$$\|u, \Omega\|_{l+k} \leq C \left(\|f, \Omega\|_{l+k-2m} + \sum_{j} \|g_j^1, \partial\Omega^1\|_{l+k-m_j^1-1/2} + \|u, \Omega\|_0 \right).$$

7. The results obtained generalize to elliptic as well as to parabolic systems.

Received
13 VI 1964

CITED LITERATURE

1. S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations under general boundary conditions, 1, IL, 1962.

2. M. S. Agranovich, A. S. Dynin, *DAN*, **146**, No. 3 (1962).
3. M. S. Agranovich, M. I. Vishik, *UMN*, **18**, No. 1 (1963).
4. Ya. B. Lopatinskii, *Ukr. Mat. Zh.*, **5**, No. 2 (1953).
5. C. Miranda, *Bull. Soc. Roy. Sci. Liège*, **31**, No. 9-10 (1962).
6. L. N. Slobodetskii, Scientific Notes, Leningrad State Pedagogical Institute named after A. I. Herzen, **197**, 54 (1958).
7. S. L. Sobolev, Some applications of functional analysis in mathematical physics, L., 1950.
8. Z. Ya. Shapiro, *Izv. AN SSSR, Ser. Mat.*, **17**, No. 6 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.