MECHANICS

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1964

SovietRxiv

Abstract

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STATISTICAL DESCRIPTION OF THE STRESSED STATE OF A DEFORMABLE BODY

(Presented by Academician A. Yu. Ishlinskii, 20 XII 1963)

When random external forces act on a solid deformable body, or when the body has a statistically specified microinhomogeneous structure, the stress tensor τ_{ij} defines a tensor random field. Using methods customary in the theory of random fields (^{1}), let us consider the question of the statistical description of the random field of the stress tensor. For simplicity, we shall carry out the consideration in a rectangular Cartesian coordinate system x_i $(i = 1, 2, 3)$.

We shall say that in some region V a tensor random field $\tau_{ij}(M)$ is specified if, to every finite system of points $M_k(x_s^k)$, $k = 1, \ldots, n$, from the region V, there is assigned an Nn-dimensional probability distribution law for the system of values $\tau_{ij}^k = \tau_{ij}(x_s^k)$, where N is the number of components of the stress tensor. We restrict ourselves to consideration of a symmetric stress tensor and introduce the $6n$-dimensional probability density

$$f^n\left(\tau_{11}^1, \tau_{12}^1, \ldots, \tau_{33}^1, \ldots, \tau_{11}^n, \tau_{12}^n, \ldots, \tau_{33}^n\right) = f^n\left(\tau_{ij}^k\right) \qquad (1)$$

for the random variables τ_{ij}^k, where the functions $f^n(\tau_{ij}^k)$, $n = 1, 2, \ldots$, satisfy certain consistency conditions (^{1}).

Let us call the quantity

$$p^{(m)}_{i_1 j_1 \ldots k_1 l_1 \ldots i_n j_n \ldots p_n r_n} =$$

$$= \overline{(\tau^1_{i_1 j_1} - \bar{\tau}^1_{i_1 j_1}) \cdots (\tau^1_{k_1 l_1} - \bar{\tau}^1_{k_1 l_1}) \cdots (\tau^n_{i_n j_n} - \bar{\tau}^n_{i_n j_n}) \cdots (\tau^n_{p_n r_n} - \bar{\tau}^n_{p_n r_n})} =$$

$$= \int_{-\infty}^{+\infty} \cdots \int (\tau^1_{i_1 j_1} - \bar{\tau}^1_{i_1 j_1}) \cdots (\tau^n_{p_n r_n} - \bar{\tau}^n_{p_n r_n}) f^n(\tau^k_{ij})\, d\tau^1_{i_1 j_1} \cdots d\tau^n_{p_n r_n}, \qquad (2)$$

the n-point moment of order m, where $\bar{\tau}_{ij}^k$ is the statistical mean of the quantity τ_{ij}^k, $m = m_1 + \cdots + m_n$; m_k is the number of factors on the right-hand side of (2) referring to the point M_k. The moment of order m (2) for the tensor field τ_{ij} is an n-point tensor (^{2}) of rank $2m$. Knowing the moments (2) of all orders, one can reconstruct the distribution functions (1) (^{1,3}); therefore a random tensor field is statistically determined by the totality of the moments (2) of all orders. We shall also say that a statistical description of the field τ_{ij} is given with accuracy up to moments of order m if, for any system of points $M_1, \dots, M_n$ ($n \leq m$), the moments (2) up to order m inclusive are known.

In what follows, the tensor field τ_{ij} will be regarded as continuous (^{1}). Then all moments (2) can be obtained by a limiting passage from moments whose type and order coincide ($n = m$); we shall call these moments the fundamental moments. Introducing the mean value σ_{ij} of the stress tensor and the deviation p_{ij} from the mean value,

$$\sigma_{ij} = \bar{\tau}_{ij}, \qquad p_{ij} = \tau_{ij} - \sigma_{ij}. \tag{3}$$

represent the principal moment of order n in the form

$$p_{i_1 j_1 \dots i_n j_n} = \overline{p_{i_1 j_1}(x_s^1) \dots p_{i_n j_n}(x_s^n)}. \tag{4}$$

The principal moment of order n is a function of the $3n$ coordinates x_s^k ($s = 1, 2, 3;\ k = 1, \dots, n$), which we shall regard as independent. In particular, for the principal moment of second order, also called the moment of connection of the field values at two points [1], we have

$$p_{ijkl} = \overline{p_{ij}(x_s^1)\, p_{kl}(x_s^2)}, \qquad p_{ijkl} = p_{jikl} = p_{ijlk} = p_{klij}. \tag{5}$$

In equilibrium of a deformable body and in the absence of body forces, the stress tensor τ_{ij} satisfies the equations

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0. \tag{6}$$

Here and below the usual summation convention for tensor quantities over repeated indices from 1 to 3 is used. We shall further apply the operation of statistical averaging (by means of the distributions (1)), which, as is known, has the properties

$$\overline{f+g} = \bar{f} + \bar{g}, \qquad \overline{f g} = \bar{f}\,\bar{g}, \qquad \frac{\partial \bar{f}}{\partial x_i} = \overline{\frac{\partial f}{\partial x_i}}, \qquad \bar{f}' = 0 \ (f' = f - \bar{f}), \tag{7}$$

where f and g are arbitrary random functions. From (6), (7), and (3) we find

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad \frac{\partial p_{ij}}{\partial x_j} = 0. \tag{8}$$

Writing the second equation (8) for the point $M_k(x_s^k)$ and using (4), we obtain $3 \cdot 6^{n-1} n$ differential equations for the principal moments

$$\frac{\partial p_{i_1 j_1 \ldots i_n j_n}}{\partial x_{j_k}^k} = 0, \qquad k = 1, \ldots, n. \tag{9}$$

For $n = 2$, relations (9) give 36 equations for the moment of connection

$$\frac{\partial p_{ijkl}}{\partial x_j^1} = 0, \qquad \frac{\partial p_{ijkl}}{\partial x_l^2} = 0. \tag{10}$$

Let us consider the special cases of statistically homogeneous and statistically isotropic fields. We shall call a random tensor field τ_{ij} statistically homogeneous if the probability distributions (1) are invariant with respect to translations $x_s^{k'} = x_s^k + a_s$. The principal moments (4) in this case depend only on $(n-1)$ vectors that determine the configuration of the points M_k. The tensor σ_{ij} (3) will then be constant, and the moment of connection (5), called in this case also the correlation tensor, will be a function of one vector ξ_s:

$$p_{ijkl} = p_{ijkl}(\xi_s), \qquad \xi_s = x_s^2 - x_s^1, \tag{11}$$

and the relation

$$p_{ijkl}(\xi_s) = p_{klij}(-\xi_s). \tag{12}$$

holds.

The equations (10) take the form

$$\frac{\partial p_{ijkl}(\xi_s)}{\partial \xi_j} = 0, \qquad \frac{\partial p_{ijkl}(\xi_s)}{\partial \xi_l} = 0, \tag{13}$$

and, by virtue of (12), only one of these two systems of equations is independent.

We shall call a random tensor field τ_{ij} statistically isotropic if it is statistically homogeneous and the probability distribution of the components of the tensor τ_{ij} in a coordinate system rigidly connected with the system of points M_k ($k = 1, \ldots, n$) is invariant with respect to rigid rotations and mirror reflections of the configuration determined by this system of points. Following Robertson' s method (^{4}) and taking (5), (12) into account, we represent the correlation tensor (11) of the statistically isotropic field τ_{ij} in the form

$$\begin{aligned} p_{ijkl}(\xi_s) = a_1\delta_{ij}\delta_{kl} &+ a_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &+ a_3(\xi_j\xi_k\delta_{il} + \xi_i\xi_l\delta_{jk} + \xi_i\xi_k\delta_{jl} + \xi_j\xi_l\delta_{ik}) \\ &+ a_4(\xi_i\xi_j\delta_{kl} + \xi_k\xi_l\delta_{ij}) + a_5\xi_i\xi_j\xi_k\xi_l, \end{aligned} \tag{14}$$

where $a_i = a_i(\rho)$ $(i = 1, \dots, 5)$, $\rho^2 = \xi_j\xi_j$, and δ_{ik} is the unit tensor of second rank.

Choosing a special coordinate system with its origin at the point M_1 and the x_1-axis passing through the point M_2, denoting the components of the tensor (11) in this system by p^*_{ijkl}, and setting

$$p^*_{1111} = p_1, \qquad p^*_{2222} = p_2, \qquad p^*_{1122} = p_3,$$

$$p^*_{2233} = p_4, \qquad p^*_{1212} = p_5, \qquad p^*_{2323} = p_6,$$

we find

$$\begin{gathered} p_1 = (a_1 + 2a_2) + 2\rho^2(2a_3 + a_4) + \rho^4 a_5, \\ p_2 = a_1 + 2a_2, \qquad p_3 = a_1 + a_4\rho^2, \qquad p_4 = a_1, \\ p_5 = a_2 + a_3\rho^2, \qquad p_6 = a_2, \end{gathered} \tag{15}$$

with the relation

$$p_4 + 2p_6 - p_2 = 0. \tag{16}$$

Solving (15) with respect to a_i, substituting them into (14), and introducing the unit vector

$$l_i = \frac{\xi_i}{\rho},$$

we obtain an analogue of the Kármán formula (^{1}), known in the theory of turbulence,

$$\begin{aligned} p_{ijkl} = p_4\delta_{ij}\delta_{kl} &+ p_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &+ (p_5 - p_6)(l_j l_k\delta_{il} + l_i l_l\delta_{jk} + l_i l_k\delta_{jl} + l_j l_l\delta_{ik}) \\ &+ (p_3 - p_4)(l_i l_j\delta_{kl} + l_k l_l\delta_{ij}) + (p_1 + p_2 - 2p_3 - 4p_5) l_i l_j l_k l_l. \end{aligned} \tag{17}$$

Putting $\rho = 0$, from (17) and (15) we have

$$p^0_{ijkl} = p_{ijkl}\big|_{\rho=0} = p_4(0)\delta_{ij}\delta_{kl} + p_6(0)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$p_1(0) = p_2(0), \qquad p_3(0) = p_4(0), \qquad p_5(0) = p_6(0),$$

$$p_4(0) + 2p_6(0) - p_2(0) = 0.$$

Substituting (17) into the first equation (13), we find

$$A_1(\rho)\xi_i\delta_{kl} + A_2(\rho)(\xi_l\delta_{ik} + \xi_k\delta_{il}) + A_3(\rho)\xi_i\xi_k\xi_l = 0, \tag{18}$$

where

$$A_1(\rho) = \frac{1}{\rho^2}\,[\rho p_3' + 2(p_5 - p_6)],$$

$$A_2(\rho) = \frac{1}{\rho^2}\,[\rho p_5' + (p_5 - p_6 + p_3 - p_4)],$$

$$A_3(\rho) = \frac{1}{\rho^4}\,[\rho(p_1' - p_3' - 2p_5') + 2(p_2 - p_3 - 2p_5)],$$

$$p_i' = \frac{dp_i}{d\rho}.$$

Since (18) holds for arbitrary ξ_j (connected only by the relation $\rho^2 = \xi_j\xi_j$), we have

$$A_i(\rho) = 0, \qquad i = 1, 2, 3,$$

which, together with (16), gives

$$\rho p_3' + 2(p_5 - p_6) = 0,$$

$$\rho p_5' + (p_5 - p_6 + p_3 - p_4) = 0,$$

$$\rho(p_1' - p_3' - 2p_5') + 2(p_2 - p_3 - 2p_5) = 0, \tag{19}$$

$$p_4 + 2p_6 - p_2 = 0.$$

Thus, the 6 functions $p_i(\rho)$, $i = 1, \dots, 6$, are connected by the 4 equations (19) and, consequently, among them there are only 2 independent ones.

Let us note in conclusion that all the results indicated above, in whose derivation the equilibrium equation (6) is not used, hold for the random field of any symmetric tensor of second rank; moreover, they can in an obvious way be generalized to the random field of a tensor of arbitrary rank.

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Received
20 XII 1963

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Note: Figure translations are in progress. See original paper for figures.

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