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## Abstract

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## MATHEMATICS

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# ON THE REPRESENTATION OF CERTAIN CLASSES OF ENTIRE AND QUASI-ENTIRE FUNCTIONS

In the present note we give formulations of several new results on the integral representation of certain general classes of analytic functions.

## A. Parametric representation of entire functions

**1°.** In the author' s papers ($^{1,2}$) certain theorems of Paley-Wiener type ($^{3}$) were established on the parametric representation of entire functions of finite order $\rho \geq 1/2$ and normal type, square-integrable in modulus along certain systems of rays of the complex plane.

These representations were constructed with the aid of entire functions of Mittag-Leffler type

$$E_\rho(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k\rho^{-1})}, \tag{1}$$

whose fine asymptotic properties formed the basis of the investigations mentioned.

A further development of the method by which these theorems were obtained has made it possible to establish a result of considerably more general nature, essentially of a final character.

Let us introduce the following notation. Let $\{\vartheta_k\}_0^{\chi+1}$ be a set of numbers

$$-\pi < \vartheta_0 < \vartheta_1 < \cdots < \vartheta_\chi \leq \pi < \vartheta_{\chi+1} = \vartheta_0 + 2\pi,$$

$$\max_{0 \leq k \leq \chi} \{\vartheta_{k+1} - \vartheta_k\} = \pi/\rho,$$

where $\rho \geq 1/2,\ \chi = [\chi] \geq [2\rho] - 1.$

Forming consecutive pairs $(\vartheta_k, \vartheta_{k+1})_0^{\chi}$, let us select from them (preserving their mutual order of succession) all those $(\vartheta_{r_k}, \vartheta_{r_{k+1}})_0^{p}$ $(0 \leq p \leq \chi)$ for which $\vartheta_{r_{k+1}} - \vartheta_{r_k} = \pi/\rho$ $(k = 0, 1, \dots, p)$, and put $\theta_k = \frac{1}{2}\{\vartheta_{r_k} + \vartheta_{r_{k+1}}\}$ $(k = 0, 1, \dots, p)$. Finally, assuming that $\omega \in (-1, +1)$ and $\sigma_k \geq 0$ $(k = 0, 1, \dots, p)$, consider the class $W_\sigma^{(\rho)}(\omega; \{\vartheta_k\}; \{\sigma_k\})$ of entire functions $f(z)$ of order $\rho$ and normal type $\leq \sigma$, satisfying the conditions:

$$1) \quad \int_0^\infty \left|f\left(te^{-i\theta_k}\right)\right|^2 t^\omega \, dt < +\infty \quad (k = 0, 1, \dots, \chi);$$

$$2) \quad h(-\theta_k; f) \leq \sigma_k \leq \sigma \quad (k = 0, 1, \dots, p),$$

where $h(\vartheta; f)$ is the indicator of the function $f(z)$.

The following fundamental theorem on the parametric representation of the class $W_\sigma^{(\rho)}(\omega; \{\vartheta_k\}; \{\sigma_k\})$ has been established.

**Theorem 1.** a) The class $W_\sigma^{(\rho)}(\omega; \{\vartheta_k\}; \{\sigma_k\})$ coincides with the set of functions $f(z)$ admitting a representation of the form

$$f(z) = \sum_{k=0}^{p} \int_0^{\sigma_k} E_\rho\{e^{i\theta_k} z t^{1/\rho}; \mu\} \varphi_k(t) t^{\mu-1} \, dt, \tag{2}$$

where $\mu = (1 + \omega + \rho)/2\rho$ and $\varphi_k(t) \in L_2(0, \sigma_k)$ $(k = 0, 1, \dots, p)$;

b) the functions $\varphi_k(\tau)$ are unique, and almost everywhere

$$\frac{i}{\sqrt{2\pi\rho}} \left\{e^{-i\frac{\pi}{2}\mu} \Phi_{r_{k+1}}(-\tau) - e^{i\frac{\pi}{2}\mu} \Phi_{r_k}(\tau)\right\} = \begin{cases} \varphi_k(\tau), & \tau \in (0, \sigma_k), \\ 0, & \tau \in (\sigma_k, +\infty), \end{cases} \quad (k = 0, 1, \dots, p) \tag{3}$$

where

$$\Phi_k(\tau) = \frac{1}{\sqrt{2\pi}} \frac{d}{d\tau} \int_0^\infty f\left(e^{-i\vartheta_k} v^{1/\rho}\right) \frac{e^{-i\vartheta_k} - 1}{-iv} v^{\mu-1} \, dv \quad (k = 0, 1, \dots, \chi). \tag{3'}$$

2°. We shall give only one of the numerous consequences following from this theorem and having independent interest.

For a given integer $p \geq 1$, denote by $C_\sigma^{(\rho)}(\omega; \{\sigma\})$ the class of entire functions $f(z)$ of order $\rho$ $(p \leq \rho < 2p)$ and of normal type $\leq \sigma$, satisfying the following conditions:

1) the integrals

$$\int_0^\infty |f(te^{-i\vartheta})|^2 t^\omega \, dt \quad (-1 < \omega < 1)$$

are finite in all intervals

$$\frac{\pi}{p}k - \delta_p \le \vartheta \le \frac{\pi}{p}k + \delta_p \quad (k = 0, 1, \dots, 2p-1),$$

where

$$\delta_p = \frac{\pi}{2}\left(\frac{1}{p} - \frac{1}{\rho}\right) < \frac{\pi}{p};$$

2)

$$h\left(-\frac{\pi}{p}\left(k + \frac{1}{2}\right); f\right) \le \sigma_k \le \sigma. \quad (k = 0, 1, \dots, 2p-1).$$

**Theorem 2.** a) The class $C_\sigma^{(\rho)}(\omega; \{\sigma_k\})$ coincides with the set of functions $f(z)$ admitting the representation

$$f(z) = \sum_{k=0}^{2p-1} \int_0^{\sigma_k} E_\rho\left\{e^{i\frac{\pi}{p}(k+\frac{1}{2})} z \tau^{1/\rho}; \mu\right\} \varphi_k(\tau) \tau^{\mu-1} \, d\tau, \tag{4}$$

where $\mu = (1 + \omega + \rho)/2\rho$ and $\varphi_k(\tau) \in L_2(0, \sigma_k)$ $(k = 0, 1, \dots, 2p-1)$;

b) the formulas

$$\frac{i}{\sqrt{2\pi\rho}} \left\{e^{-i\frac{\pi}{2}\mu} \psi_{k+1}^{(-)}(-\tau) - e^{i\frac{\pi}{2}\mu} \psi_k^{(+)}(\tau)\right\} = \begin{cases} \varphi_k(\tau), & \tau \in (0, \sigma_k), \\ 0, & \tau \in (\sigma_k, +\infty), \end{cases} \quad (k = 0, 1, \dots, 2p-1) \tag{5}$$

are valid, where

$$\psi_k^{(\pm)}(\tau) = \frac{1}{\sqrt{2\pi}} \frac{d}{d\tau} \int_0^\infty \frac{e^{-i\tau v} - 1}{-iv} f\left(e^{-i\left(\frac{\pi}{p}k + \delta_p\right)} v^{1/\rho}\right) v^{\mu-1} \, dv \quad (k = 0, 1, \dots, 2p-1). \tag{6}$$

Let us note that, in particular, for $\rho = p = 1$ and $\omega = 0$, this theorem yields the classical Paley–Wiener theorem on the representation of entire functions of exponential type belonging to the class $L_2(-\infty, +\infty)$.

**B. Representation of functions analytic on the Riemann surface of the logarithm.**

1°. In a joint work of A. E. Avetisyan and the author [4], a representation was established for functions analytic in the angle

$$\Delta(\alpha) : \{|\operatorname{Arg} z| < \pi/2\alpha, \ 0 < |z| < \infty\} \quad (1/2 < \alpha < \infty),$$

of opening $\pi/\alpha < 2\pi$, and possessing

with a prescribed finite growth. This representation was constructed by means of a special contour integral transformation with a kernel of the form $E_\rho(z\zeta;\mu)$.

Analogous representations can be established for analytic functions of finite growth, defined in an angle of type $\Delta(\alpha)$, but of arbitrary opening $\pi/\alpha$ $(0 < \alpha < \infty)$, lying on the Riemann surface of the logarithmic function, i.e. in the domain $G_\infty : \{-\infty < \operatorname{Arg} z < +\infty,\ 0 < |z| < \infty\}$. However, in this case the representations obtained by us are integral transformations along special contours lying on $G_\infty$, and with a kernel of the form $v_\rho(z\zeta;\mu)$, where, by definition,

$$v_\rho(z;\mu) = \int_0^\infty \frac{z^t}{\Gamma(\mu + t/\rho)}\, dt. \tag{7}$$

Let us note that the function $v_\rho(z;\mu)$, being, obviously, a continual analogue of the function $E_\rho(z;\mu)$, is connected with the well-known Volterra function

$$v(z;\mu) = \int_0^\infty \frac{z^{t+\mu}}{\Gamma(1+\mu+t)}\, dt \tag{7'}$$

by the formula

$$v_\rho(z;\mu) = \rho z^{\rho(1-\mu)} v\left(z^\rho;\mu-1\right). \tag{8}$$

For this reason, in establishing the two main theorems given below, we relied substantially on the important asymptotic properties of the function $v(z;\mu)$, investigated in the author’ s paper (5).

We introduce several preliminary notations. Let $D_\rho(\vartheta;\nu)$ $(0 < \rho < \infty,\ -\infty < \vartheta < \infty,\ 0 \leqslant \nu < \infty)$ be the unbounded domain $\operatorname{Re}(e^{-i\vartheta}\zeta)^\rho > \nu,\ |\operatorname{Arg}\zeta - \vartheta| < \pi/2\rho$ with boundary $L_\rho(\vartheta;\nu) : \{(e^{-i\vartheta}\zeta)^\rho = \nu,\ -\infty < \tau < +\infty\}$, lying on the surface $G_\infty$.

The union of the domains $\{D_\rho(\vartheta;\nu)\}$ over all values of the parameter $\vartheta \in [-\pi/2\alpha, \pi/2\alpha]$ will be denoted by $D^{(\alpha)}(\nu)$. The contour $L_\rho^{(\alpha)}(\nu)$ of the unbounded domain $D^{(\alpha)}(\nu) \in G_\infty$ consists of the arc $-\pi/2\alpha \leqslant \operatorname{Arg}\zeta \leqslant \pi/2\alpha$, $|\zeta| = \nu^{1/\rho}$, and of the unbounded curves beginning at its endpoints

$$L_\rho^{(\pm)}\left(\pm\frac{\pi}{2\alpha};\nu\right) : (e^{\pm i\pi/2\alpha}\zeta)^\rho = \nu \pm i\tau, \quad 0 \leqslant \tau < +\infty.$$

Finally, denote by $A^{(\alpha)}[\rho_1,\sigma_1]$ $(0 < \alpha < \infty;\ 0 \leqslant \sigma_1 < \infty;\ 0 < \rho_1 < \infty)$ the class of functions $F(z)$, analytic in the domain $\Delta(\alpha) \in G_\infty$, for which the estimate $|F(z)| \leqslant M_F e^{\sigma_1 |z|^{\rho_1}},\ z \in \Delta(\alpha)$, holds.

The integral representation of the class $A^{(\alpha)}[\rho_1,\sigma_1]$ is given by the theorem:

**Theorem 3.** If $F(z) \in A^{(\alpha)}[\rho_1, \sigma_1]$, then for every $\rho \geqslant \rho_1$ the following two assertions hold:

a) for each $\vartheta \in [-\pi/2\alpha, \pi/2\alpha]$ the formula

$$g_\rho(\zeta; F) = \rho\,(e^{-i\vartheta}\zeta)^{\mu\rho}\zeta^{-1} \int_0^\infty F\left(te^{-i\vartheta}\right) e^{-t^\rho (e^{-i\vartheta}\zeta)^\rho} t^{\mu\rho - 1}\, dt, \quad \zeta \in D_\rho(\vartheta; \nu)\ (\mu > 0) \tag{9}$$

defines a function analytic in the domain $D_\rho^{(\alpha)}(\nu_0)$, where $\nu_0 = \sigma_1$ for $\rho = \rho_1$ and $\nu_0 = 0$ for $\rho > \rho_1$;

b) the integral formula holds

$$F(z) = \frac{1}{2\pi i} \int_{L_\rho^{(\alpha)}(\chi)} v_\rho(z\zeta; \mu)\, g_\rho(\zeta; F)\, d\zeta, \quad z \in \Delta(\alpha), \tag{10}$$

for any $\chi > \nu_0$ and $\mu \in (0, 1/2]$.

2°. We shall call a function $f(z)$ **quasi-entire** if it is regular on the whole Riemann surface $G_\infty$, except for its branch points $z = 0$ and $z = \infty$, and if, for any way of tending with $z \in G_\infty$ to the point $z = 0$, there exists a finite limit

$$f(0) = \lim_{z \to 0} f(z).$$

We assign to the class $C_{(\rho,\sigma)}$ quasi-entire functions $f(z)$ satisfying the conditions:

1)
$$M_f(r) = \sup_{-\infty < \vartheta < \infty} |f(re^{i\vartheta})| < +\infty, \qquad 0 < r < \infty;$$

2) the order of the function $M_f(r)$ is not greater than $\rho$, i.e.

$$\overline{\lim}_{r \to \infty} (\log r)^{-1} \log_2 M_f(r) \leq \rho,$$

and, in the case of equality, we also have

$$\overline{\lim}_{r \to \infty} r^{-\rho} \log M_f(r) \leq \sigma.$$

As follows easily from the asymptotics of the function $\nu(z; \mu)$, the simplest and most important example of a function of the class $C_{(\rho,\sigma)}$ is the function $\nu_\rho(z\xi; \mu)$ for any $\rho > 0$, $\mu \in (0, +\infty)$, and $|\xi| = \sigma^{1/\rho}$ ($^5$).

For each function $f(z) \in C_{(\rho,\sigma)}$ and for any $\vartheta \in (-\infty, +\infty)$, one can define its Borel-type transform $g_\rho(\zeta; f)$ according to formula (9). It is analytic on the entire Riemann surface $G_\infty(\chi) : \{-\infty < \operatorname{Arg}\zeta < +\infty,\ \chi < |\zeta| < \infty\}$, where

$\chi = \sigma^{1/\rho}$ if the function $M_f(r)$ has order $\rho$ and type $\sigma$, and $\chi = 0$ if the order or the type of this function is lower than $\rho$ or $\sigma$.

We note that in the work of A. Pfluger (⁶) a formula was proposed which makes it possible to reconstruct a function $f(z) \in C_{(1,\sigma)}$ by means of its Borel transform $g_1(\zeta; f)$, analogously to Pólya' s well-known theorem for entire functions of exponential type.

But the indicated formula of A. Pfluger has the essential shortcoming that, being an integral transform of the function $g_1(\zeta; f)$ with kernel $e^{z\zeta}$ along a special contour depending on a parameter $\Phi \in (-\infty, \infty)$, it represents the function $f(z)$ not on the whole surface $G_\infty$, but only in the corresponding half-plane

$$\operatorname{Re}(e^{-i\Phi} z) > 0.$$

A natural and complete solution of the problem of representing quasi-entire functions of the class $C_{(\rho,\sigma)}$ is given by the following theorem.

**Theorem 4.** *If $f(z) \in C_{(\rho,\sigma)}$ and $g_\rho(\zeta; f)$ is its Borel-type transform, then for any $\tau > \chi$ and $\mu \in (0, \frac{1}{2}]$ the integral formula is valid*

$$f(z) = \frac{1}{2\pi\rho i} \int_{-\infty}^{\infty} \nu'_\rho(\tau z e^{i\vartheta}; \mu)\, g_\rho(\tau e^{i\vartheta}; f)\, d(\tau e^{i\vartheta}), \qquad z \in G_\infty. \qquad (11)$$

In conclusion, we note that Theorems 3 and 4 may be regarded as peculiar approximation theorems for functions of the classes $A^{(\alpha)}[\rho_1, \sigma_1]$ and $C_{(\rho,\sigma)}$ by quasi-entire functions of the simplest nature, namely the function $\nu_\rho(z\xi; \mu)$.

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*Note: Figure translations are in progress. See original paper for figures.*