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## Abstract

## Full Text

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### MATHEMATICAL PHYSICS

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# ON CONDITIONS FOR THE ANALYTICITY OF THE SCATTERING MATRIX

*(Presented by Academician V. A. Fock on 21 III 1964)*

In this paper it is proved that the Jost function $f(k)$ is an entire function of the complex variable $k$ of growth order not exceeding one and such that $f(k) - 1$ is quadratically integrable in $k$ if and only if the potential is finite. Thus it is clarified that the class of potentials considered by Regge ([1]) is the maximal admissible one for the function $f(k)$ to be entire of growth order not exceeding one.

Let us consider the Schrödinger equation

$$y'' + [k^2 - p(x)]y = 0 \tag{1}$$

under the assumption on the potential

$$\int_0^\infty x|p(x)|\, dx < \infty. \tag{2}$$

Consider the solution of equation (1) defined by the following asymptotics for large $x$:

$$\lim_{x\to\infty} e^{-ikx} f(x, k) = 1. \tag{3}$$

The function $f(k) = f(0, k)$ is called the Jost function. The basic properties of solutions of equation (1) are collected in the book ([2]). The analytic properties of the function $f(k)$, under the assumption that the potential $p(x)$ is finite, were established in ([1,3]).

Our task is to invert some of the results obtained by Regge. In this direction we shall prove the theorem:

**Theorem 1.** *Let $f(k)$ be an entire function of growth order not exceeding one, and let $f(k)-1$ be quadratically integrable. Suppose that $f(k)$ is the Jost function of some scattering problem of the type (1)–(3), $M_n$ being normalization constants.** 
*Then the potential $p(x)$ is finite.*

**Proof.** We shall base the proof on the Wiener–Paley theorem ([4]) and on the properties of transformation operators ([2]). It is shown in ([2]) that

$$f(x,k) = e^{ikx} + \int_x^\infty A(x,y)e^{iky}\,dy. \tag{4}$$

From (4) we obtain for $f(k)$ the formula

$$f(k) = 1 + \int_0^\infty A(0,y)e^{iky}\,dy. \tag{5}$$

* Specially chosen; see the note to formula (13).

The function $A(x,y)$ satisfies the equation of V. A. Marchenko ([5]):

$$A(x,y) = F(x+y) + \int_x^\infty A(x,t)F(t+y)\,dt, \qquad x \le y, \tag{6}$$

where

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[-S(k)+1]e^{ikt}\,dk + \sum_{n=1}^{m} M_n^2 e^{-k_n t}, \tag{7}$$

$$S(k) = \frac{f(-k)}{f(k)}; \tag{8}$$

the points $k = ik_n,\ k_n > 0$, are poles of the function $f(k)$.

For us the essential relation between the potential $p(x)$ and the function $A(x,y)$ is:

$$p(x) = -2\frac{d}{dx}A(x,x). \tag{9}$$

All these formulas are contained in the book ([2]).

From (6), for $x = 0$, we obtain the equation

$$A(y) = F(y) + \int_0^\infty A(t)F(t+y)\,dt, \qquad y \ge 0, \tag{10}$$

where

$$A(y) = A(0, y). \tag{11}$$

By virtue of the Wiener–Paley theorem, from the conditions imposed on $f(k)$ and from equation (5) it follows that $A(y)$ is a finite function, square-integrable.

Transform (10) to the form

$$A(y) = F(y) + \int_y^\infty A(v-y)F(v)\,dv, \qquad y \geq 0. \tag{12}$$

We shall show that from equation (12) it follows that $F(y) \equiv 0$ for sufficiently large $y$.

Equation (12) is a Volterra equation with a difference kernel. It can be shown (see the Appendix) that there exists a solution of equation (12) having the form*:

$$F(y) = A(y) + \int_y^\infty \Gamma(v-y)A(v)\,dv. \tag{13}$$

Since the function $A(y)$ is finite, $A(y) = 0$ for $y > y_0$. From (13) it follows that for $y > y_0$, $F(y) = 0$.

To prove the finiteness of the potential we use formula (9) and the following equation, which follows from (6):

$$A(x, x) = F(2x) + \int_x^\infty A(x, t)E(t + x)\,dt. \tag{14}$$

From equation (14) we conclude that for $2x > y_0$, $A(x, x) = 0$. Hence, from (9), we obtain the finiteness of $p(x)$. Theorem 1 is proved.

**Appendix.** Here we shall consider the question of the solvability

* With a suitable choice of the constants $M_n$. If the discrete spectrum is absent, then all $M_n = 0$, and the reasoning can be simplified.

equations of type (12). Let

$$g(x) = f(x) + \int_x^\infty A(y-x)f(y)\,dy. \tag{15}$$

Let, for some $c > 0$, $e^{cx}g(x) \subset L_2(0,\infty)$, $e^{cx}A(x) \subset L(0,\infty)$.

**Theorem 2.** *Under the assumptions made, there exists a unique solution of equation (15) such that $e^{c'x} fx \subset L_2(0,\infty)$, $0 < c'$. This solution has the form*

$$f(x) = g(x) + \int_x^\infty \Gamma(y-x) g(y)\, dy, \tag{16}$$

where $\Gamma$ is some function.

**Proof.** Multiply both sides of equation (15) by $e^{bx}$, $b > 0$. We obtain the equation

$$g_b(x) = f_b(x) + \int_x^\infty A_b(y-x) f_b(y)\, dy, \tag{17}$$

where, for example, $g_b(x) \equiv g(x)e^{bx}$.

Introduce the functions:

$$G(x) = \begin{cases} g_b(x), & x \geq 0, \\ 0, & x < 0; \end{cases} \tag{18a}$$

$$E(x) = \begin{cases} f_b(x), & x \geq 0, \\ f_b(x), & x < 0, \end{cases} \tag{18 }$$

$$B(x) = \begin{cases} A_b(x), & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{18 }$$

Equation (17) can be rewritten in the form

$$G(x) = F(x) + \int_{-\infty}^\infty B(y-x) F(y)\, dy. \tag{19}$$

Take the Fourier transform of (19). We obtain[6]

$$\widetilde{G}(k) = \widetilde{F}(k) + \widetilde{B}(-k)\widetilde{F}(k). \tag{20}$$

Hence

$$\widetilde{F}(k) = \frac{\widetilde{G}(k)}{1+\widetilde{B}(-k)} = \widetilde{G}(k) - \frac{\widetilde{B}(-k)\widetilde{G}(k)}{1+\widetilde{B}(-k)}. \tag{21}$$

Choosing $b > 0$ sufficiently large, we can ensure that $1 + \widetilde{B}(-k) \neq 0$ for $-\infty < k < \infty$. Then, by a well-known theorem of Wiener[5],

$$\frac{1}{1+\widetilde{B}(-k)} = \widetilde{\mathcal{L}}(-k),$$

where $\widetilde{\mathcal{L}}(-k)$ is the Fourier transform of some function from $L(0, -\infty)$. Thus,

$$\widetilde{F}(k) = \widetilde{G}(k) + \tilde{\Gamma}(-k)\widetilde{G}(k), \tag{22}$$

where $\tilde{\Gamma}(-k) = \widetilde{B}(-k)\widetilde{\mathcal{L}}(-k)$ is the Fourier transform of some function from $L(0, -\infty)$. Inverting (22) and using the convolution theorem, we arrive at the formula:

$$F(x) = G(x) + \int_{-\infty}^{\infty} \Gamma(y-x)G(y)\,dy. \tag{23}$$

But for $x < 0$, $\Gamma(x) \equiv 0$. This follows from a theorem of Titchmarsh (([6], 172), if one takes into account that the function

$$\tilde{\Gamma}(-k) = \frac{\widetilde{B}(-k)}{1+\widetilde{B}(-k)}$$

is analytic in the half-plane $\operatorname{Im} k \leqslant 0$, together with the functions $\widetilde{B}(-k)$ and

$$\frac{1}{1+\widetilde{B}(-k)}.$$

(Recall that $\widetilde{B}(-k)$ is an entire function of the complex variable $k$, since $f(k)$ is, by assumption, an entire function.) Therefore (23) has the form (16). The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*