SovietRxiv

Soviet-era science, translated into English

A PARTICULAR SOLUTION OF THE EQUATIONS OF THE THEORY OF IDEAL PLASTICITY IN CYLINDRICAL COORDINATES

1964

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196401.68062

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

THEORY OF ELASTICITY

M. A. ZADOYAN

A PARTICULAR SOLUTION OF THE EQUATIONS OF THE THEORY OF IDEAL PLASTICITY IN CYLINDRICAL COORDINATES

(Presented by Academician L. I. Sedov on 6 II 1964)

The equations of the theory of ideal plasticity under the Huber–Mises plasticity criterion in cylindrical coordinates have the form:

$$\begin{gathered} \frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0, \\ \frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0, \\ \frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0; \end{gathered} \tag{1}$$

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\left(\tau_{r\theta}^2+\tau_{\theta z}^2+\tau_{rz}^2\right)=6k^2; \tag{2}$$

$$\begin{gathered} \varepsilon_r=\frac{\partial u}{\partial r}=\lambda(2\sigma_r-\sigma_\theta-\sigma_z), \\ \varepsilon_\theta=\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}=\lambda(2\sigma_\theta-\sigma_z-\sigma_r), \\ \varepsilon_z=\frac{\partial w}{\partial z}=\lambda(2\sigma_z-\sigma_r-\sigma_\theta); \end{gathered} \tag{3}$$

$$\begin{gathered} 2\gamma_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}=6\lambda\tau_{r\theta}, \\ 2\gamma_{\theta z}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial\theta}=6\lambda\tau_{\theta z}, \\ 2\gamma_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}=6\lambda\tau_{rz}. \end{gathered} \tag{4}$$

From relations (3)–(4) we have

SovietRxiv

$$u(r,\theta,z) = u_0(r,z) - \int_0^\theta \left(r\frac{\partial v}{\partial r} - v \right) d\theta + 2r\int_0^\theta \gamma_{r\theta}\, d\theta; \tag{5}$$

$$v(r,\theta,z) = v_0(r,z) - \int_0^\theta u\, d\theta - r\int_0^\theta (\varepsilon_r + \varepsilon_z)\, d\theta; \tag{6}$$

$$w(r,\theta,z) = w_0(r,z) - r\int_0^\theta \frac{\partial v}{\partial z}\, d\theta + 2r\int_0^\theta \gamma_{\theta z}\, d\theta, \tag{7}$$

where $u_0,\ v_0,\ w_0$ are arbitrary functions of the coordinates r and z.

Assuming that the strain-rate tensor does not depend on r and z, we obtain

$$\varepsilon_r = \frac{\partial u_0}{\partial r} - r\frac{\partial^2 v_0}{\partial r^2}\theta + 2\int_0^\theta \gamma_{r\theta}\, d\theta = A_0 + A_1\theta + 2\int_0^\theta \gamma_{r\theta}\, d\theta; \tag{8}$$

$$\gamma_{rz} = \frac{1}{2}\left(\frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial r} \right) - r\frac{\partial^2 v_0}{\partial r\, \partial z}\theta + \int_0^\theta \gamma_{\theta z}\, d\theta = C_0 + C_1\theta + \int_0^\theta \gamma_{\theta z}\, d\theta; \tag{9}$$

$$\varepsilon_z = \frac{\partial w_0}{\partial z} - r\int_0^\theta \frac{\partial^2 v}{\partial z^2}\, d\theta, \tag{10}$$

where A_0, A_1, C_0, C_1 are arbitrary constants. From the preceding relations it follows that

$$\frac{\partial^2 v}{\partial z^2} + \int_0^\theta d\theta \int_0^\theta \frac{\partial^2 v}{\partial z^2}\, d\theta = \frac{\partial^2 v_0}{\partial z^2} - \frac{\partial^2 u_0}{\partial z^2}\theta, \tag{11}$$

and, consequently,

$$\frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v_0}{\partial z^2}\cos\theta - \frac{\partial^2 u_0}{\partial z^2}\sin\theta. \tag{12}$$

Substituting (12) into (10), we find

$$\varepsilon_z = \frac{\partial w_0}{\partial z} + r\frac{\partial^2 u_0}{\partial z^2} - r\frac{\partial^2 v_0}{\partial z^2}\sin\theta - r\frac{\partial^2 u_0}{\partial z^2}\cos\theta = B_0 + B_1\sin\theta + B_2\cos\theta; \quad B_0, B_1, B_2 = \text{const.} \tag{13}$$

Comparison of relations (8), (9), and (13) leads to the conclusion that the arbitrary constants B_1, B_2, C_1 are equal to zero and to the formulas

SovietRxiv

$$u_0(r,z) = A_0 r + G_1 z + G_0; \tag{14}$$

$$v_0(r,z) = (A_1 + D_1)r - A_1 r \ln r + E_1 z + E_0; \tag{15}$$

$$w_0(r,z) = (2C_0 - G_1)r + B_0 z + D_0, \tag{16}$$

where $D_1, D_0, E_1, E_0, G_1, G_0$ are new arbitrary constants. Determining from (6) the value of $r\,\partial v/\partial r - v$ and substituting into (5), we obtain

$$u + \int_0^\theta d\theta \int_0^\theta u\,d\theta = u_0 - \left(r\frac{\partial v_0}{\partial r} - v_0\right)\theta + r\int_0^\theta d\theta \int_0^\theta \varepsilon_r\,d\theta + 2r\int_0^\theta \gamma_{r\theta}\,d\theta. \tag{17}$$

Hence

$$u(r,\theta,z) = (A_0 + A_1\theta)r + (E_1 z + E_0)\sin\theta + (G_1 z + G_0)\cos\theta + 2r\int_0^\theta \gamma_{r\theta}\,d\theta. \tag{18}$$

Substitution of (18) into (6) gives

$$\begin{aligned} v(r,\theta,z) &= (A_1 + D_1)r - A_1 r\ln r - (2A_0 + B_0)r\theta - A_1 r\theta^2 \\ &\quad + (E_1 z + E_0)\cos\theta - (G_1 z + G_0)\sin\theta - 4r\int_0^\theta \gamma_{r\theta}(\xi)(\theta - \xi)\,d\xi. \end{aligned} \tag{19}$$

Determining from (6) and (18) that $\partial v/\partial z = E_1\cos\theta - G_1\sin\theta$, we shall have

$$w(r,\theta,z) = 2C_0 r + B_0 z + D_0 - (E_1\sin\theta + G_1\cos\theta)r + 2r\int_0^\theta \gamma_{r\theta}\,d\theta. \tag{20}$$

With the aid of relations (3)–(4), the stress components are represented in the form

$$\sigma_r = \sigma_\theta + \frac{2\varepsilon_r + \varepsilon_z}{\gamma_{r\theta}}\tau_{r\theta}, \qquad \sigma_z = \sigma_\theta + \frac{\varepsilon_r + 2\varepsilon_z}{\gamma_{r\theta}}\tau_{r\theta}, \qquad \tau_{rz} = \frac{\gamma_{rz}}{\gamma_{r\theta}}\tau_{r\theta}. \tag{21}$$

Substituting (21) into the equilibrium equations (1) and assuming that $\tau_{r\theta}$ does not

SovietRxiv

depends on r and z, we obtain

$$r\frac{\partial \sigma_\theta}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \sigma_r - \sigma_\theta = 0,$$

$$\frac{\partial \sigma_\theta}{\partial \theta} + 2\tau_{r\theta} = 0, \qquad r\frac{\partial \sigma_\theta}{\partial z} + \frac{\partial \tau_{\theta z}}{\partial \theta} + \tau_{rz} = 0. \tag{22}$$

From the second equation (22) we determine

$$\sigma_\theta(r, \theta, z) = H(r, z) - 2\int_0^\theta \tau_{r\theta}(\theta)\, d\theta, \tag{23}$$

where H is an arbitrary function of r and z. Substituting the result into the first and third of equations (22), we obtain $H = M \ln r + N$, and

$$\frac{d\tau_{r\theta}}{d\theta} + \frac{2\varepsilon_r + \varepsilon_z}{\gamma_{r\theta}}\tau_{r\theta} + M = 0, \qquad \frac{d\tau_{\theta z}}{d\theta} + \frac{\gamma_{rz}}{\gamma_{\theta z}}\tau_{\theta z} = 0, \tag{24}$$

where M and N are arbitrary constants. From equations (24) and the yield condition (2) the following formulas result:

$$\gamma_{r\theta} = \frac{\sqrt{3}}{2} B_0 \frac{\tau_{r\theta}}{\Omega}, \qquad \gamma_{\theta z} = \frac{\sqrt{3}}{2} B_0 \frac{\tau_{\theta z}}{\Omega}, \tag{25}$$

$$\Omega(\theta) = \sqrt{k^2 - \tau_{r\theta}^2 - \tau_{\theta z}^2 - \frac{1}{4}(\tau_{r\theta}' + M)^2 - \tau_{rz}^2}. \tag{26}$$

Using relations (8), (9), (13), and (25), from (24) we obtain

$$B_0 \Omega \frac{d}{d\theta}\left[\frac{1}{\Omega}\left(\frac{d\tau_{r\theta}}{d\theta} + M\right)\right] + \frac{4A_1}{\sqrt{3}}\Omega + 2B_0\tau_{r\theta} = 0, \tag{27}$$

$$\Omega \frac{d}{d\theta}\left[\frac{1}{\Omega}\frac{d\tau_{\theta z}}{d\theta}\right] + \tau_{\theta z} = 0. \tag{28}$$

We also have

$$\tau_{r\theta}'(0) = -M - (2A_0 + B_0)\sqrt{\frac{k^2 - Q_1^2 - Q_2^2}{A_0^2 + A_0 B_0 + B_0^2 + C_0^2}},$$

$$\tau_{\theta z}'(0) = -C_0\sqrt{\frac{k^2 - Q_1^2 - Q_2^2}{A_0^2 + A_0 B_0 + B_0^2 + C_0^2}}, \tag{29}$$

SovietRxiv

where $Q_1 = \tau_{r\theta}(0)$ and $Q_2 = \tau_{\theta z}(0)$ are also arbitrary constants.

Thus, the problem has been reduced to the construction of solutions of a system of two ordinary differential equations. The solution obtained contains 14 arbitrary constants. From it, by taking $\tau_{\theta z} = \tau_{rz} = B_0 = 0$, we obtain the known case of plane deformation of a plastic mass between inclined rough plates, investigated by A. Nadai ([1]). Setting $A_1 = D_1 = E_1 = G_1 = G_0 = E_0 = 0$, we obtain certain cases of spatial flow of a plastic mass between inclined rigid plates, when the plates rotate with a prescribed velocity about the line of intersection of the contact surfaces. The boundary conditions, analogously to ([7]), are satisfied in the integral sense. A somewhat different particular solution in cylindrical coordinates, obtained in this way, is found in the article ([8]).

Other particular solutions of spatial problems in cylindrical coordinates having a different structure are given in works ([2–6]).

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

Received
3 II 1964

CITED LITERATURE

[1] A. Nadai, *Zs. Phys.*, **30** (1924).
[2] A. Yu. Ishlinskii, *Prikl. matem. i mekh.*, **8**, issue 3 (1944).
[3] R. Hill, *The Mathematical Theory of Plasticity*, Oxford, 1950.
[4] R. T. Shield, *Proc. Roy. Soc.*, A **233**, 267 (1955).
[5] D. D. Ivlev, *Prikl. matem. i mekh.*, **22**, issue 5 (1958).
[6] D. D. Ivlev, *DAN*, **123**, No. 6 (1958).
[7] M. A. Zadoyan, *Izv. AN ArmSSR, ser. fiz.-matem. nauk*, No. 3 (1964).
[8] M. A. Zadoyan, *Dokl. AN ArmSSR*, **39**, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.