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1964

SovietRxiv

Abstract

Full Text

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On the Zeros of the Gradient of a Harmonic Function

(Presented by Academician M. A. Lavrent' ev, 20 IV 1964)

In this note we consider the structure of the set N_h of points of a domain D of three-dimensional Euclidean space R^3 at which the gradient of a harmonic function h, regular in the domain D, vanishes. The main result of the note is contained in Theorem 2. The study of the set N_h is essentially based on certain results of the paper ([1]). We formulate here the theorem from ([1]) that we need (Theorem 1 of the present note). Before formulating this theorem, it is necessary to give a number of definitions and state a number of facts concerning the properties of real-analytic sets.

Definition. Let B be an open set in the space R^3. A subset E of the set B is called **real-analytic** at a point X if there exists an open neighborhood U of the point X such that the set $E \cap U$ coincides with the zero set of some function f, real-analytic in the neighborhood U. The set E is called **analytic in the open set** B if it is analytic at all points of the set B ([1]).

Since an analytic set E is a subset of the Euclidean space R^3, it can be endowed with the induced topology; that is, the **open subsets** W of the set E are the sets $U \cap E$, where U is an open set of the space R^3. Everywhere below we shall regard E as endowed precisely with this topology.

It follows from the definition of a real-analytic set that a set E real-analytic in an open set B is always closed relative to the set B.

A set E, analytic in an open set U of the space R^3, is called **irreducible in** U if, from a representation of the set E in the form of a union $E_1 \cup E_2$ of two analytic sets E_1 and E_2, it follows that either $E = E_1$ or $E = E_2$.

A set E, analytic in an open set B of the space R^3, is called **irreducible at the point** X if there exists an open neighborhood U of the point X in B such that in every open neighborhood $V \subset U$ of the point X in B the set $E \cap V$ is irreducible in V as a real-analytic set.

We shall say that an analytic set E, irreducible at a point $X \in B$, has dimension p at the point X if there exists an open neighborhood U of the point X in B such that, for every open neighborhood $V \subset U$ of the point X in B,

$$\dim E \cap V = p.$$

It is known ([1]) that a set E, analytic at a point X, can be represented uniquely (up to order) as a union of some finite number of analytic sets E_i irreducible at the point X.

Let us consider the local structure of real-analytic sets. Denote by $V_p(E)$ the set of those points of E at which E is a real-analytic manifold of dimension p (that is, every point of the set $V_p(E)$ has a neighborhood homeomorphic to a p-dimensional ball of Euclidean space, the homeomorphism being effected by real-analytic functions ([2])). It is known ([3]) that if the set—

the set E at the point X has dimension p, then there exist a neighborhood U of the point X, open in B, and an analytic set S of dimension $< p$ such that

$$S \subset E \cap U = (V_p(E) \cap U) \cup S. \tag{1}$$

Relation (1) shows that in every p-dimensional analytic set E there exist points at which E is a real-analytic manifold of dimension p. Consequently, it follows from relation (1) that every real-analytic set of dimension p contains a set homeomorphic to a p-dimensional ball of Euclidean space.

The following results are known ([4]) concerning the structure of real-analytic sets of dimension p that are irreducible at the point X: in a neighborhood of the point $X \in R^3$ there exist local coordinates x_1, x_2, x_3 in R^3 (the coordinates of the point X are the numbers $0, 0, 0$) and a distinguished irreducible polynomial $P(x_{p+1}; x_1, \dots, x_p)$, $p \leq 2$, with real values for real $x_1, \dots, x_{p+1}$, vanishing on E and possessing the following properties: in the set defined by the equation $P = 0$, the coordinates of a point of E are analytic functions of $x_1, \dots, x_{p+1}$ (the polynomial is called distinguished if it has the form $P = x_{p+1}^q + a_1(x_1, \dots, x_p)x_{p+1}^{q-1} + \dots + a_q(x_1, \dots, x_p)$, $a_i(0, \dots, 0) = 0$, $i = 1, \dots, q$, where all $a_i(x_1, \dots, x_p)$, $i = 1, \dots, q$, are analytic functions of $x_1, \dots, x_p$). If Δ denotes the discriminant of the polynomial P, then every point of E at which $\Delta \neq 0$ belongs to $V_p(E)$, and the set of these points is dense in $V_p(E)$.

Since every analytic set at the point X is representable as the union of a finite number of analytic sets irreducible at the point X, it is enough to consider the structure of analytic sets irreducible at the point X. For sets irreducible at the point in (1) the following fact is proved.

Theorem 1. *Let E be a real-analytic set in an open set B in R^3, irreducible of dimension p at the point X. There exists a neighborhood U of the point X, open in B, such that $U \cap V_p(E)$ has a finite number of connected components A_i, and the point X is adherent to each A_i (i.e. there exists an arc γ with endpoint at the point X and $\gamma - \{X\} \subset A_i$).*

We proceed to the study of the zeros of the gradient of a harmonic function h in a domain D of three-dimensional Euclidean space R^3. Denote by N_h the set of points $X \in D$ such that $\operatorname{grad} h(X) = 0$. In what follows, by a **narrow analytic curve** we shall mean a connected real-analytic manifold of dimension 1.

Lemma. *If $h \not\equiv$ const, then the dimension of the set N_h does not exceed 1.*

Proof. The set N_h is an analytic subset of the domain D of three-dimensional Euclidean space R^3, since it is the set of zeros of the analytic function $h_x^2+h_y^2+h_z^2$. Suppose that the dimension of the set N_h is equal to 3; then, as follows from relation (1), the set N_h contains an open subset of R^3, and therefore $h_x \equiv h_y \equiv h_z \equiv 0$, whence $h \equiv$ const. Suppose that the dimension of N_h is equal to 2. In the set N_h, as follows from relation (1), there is a point X such that it has a neighborhood W homeomorphic to a disk; moreover, at every point belonging to W there exists a normal to N_h, since W is an open set of a real-analytic manifold. Since, by the condition on N_h, $\operatorname{grad} h = 0$, we have

$$h|_W = \text{const}, \qquad \frac{\partial h}{\partial n}\bigg|_W = 0,$$

where n is the normal to N_h. By the Cauchy–Kovalevskaya theorem ([5]), $h \equiv$ const. Thus, if $h \not\equiv$ const, then the dimension of the set N_h is less than D.

Theorem 2. Let D be a bounded domain in the space R^3; let h be a harmonic function, regular in the domain D; let N_h be the set of points of D at which $\operatorname{grad} h = 0$; and let K be a compact subset of the domain D. The set $N_h \cap K$ consists of a finite number of isolated points and a finite number of pieces of analytic curves.

Proof. The set N_h is a one-dimensional real-analytic set; therefore it may contain nonempty components of dimensions 0 and 1.

Let $X \in N_h$. In some neighborhood U of the point X, the set N_h can be represented as the union of a finite number of irreducible analytic sets in U, E_i ($i = 1, \dots, r$), of dimensions 0 and 1. Suppose that among the sets E_i there are r_0 zero-dimensional sets. Each zero-dimensional set is a point, since it is irreducible.

Consider the one-dimensional components E_j ($j = 1, \dots, r_1;\ r_1 \le r$) at the point X of the set N_h. Two cases are possible: a) the set E_j is a real-analytic manifold at the point X; then there exists a neighborhood U_j of the point X such that $E_j \cap U_j$ is a piece of an analytic curve; b) E_j is not a real-analytic manifold; then, by Theorem 1, there exists a neighborhood U_j of the point X such that $E_j \cap U_j$ is the union of a finite number of pieces of analytic curves issuing from the point X. $E_j \cap U_j$ coincides with the set of zeros of the indicated polynomial

$$x_2^q + a_1(x_1)x_2^{q-1} + \dots + a_q(x_1).$$

If q is even, then, since the coefficients $a_i(x_1)$ are real, the number of these curves is even. If q is odd, then for all x_1 the polynomial has at least one real root; to each such root there correspond two curves issuing from the point X. Consequently, the number of these curves is always even.

Denote by V the intersection

$$\bigcap_{j=1}^{r} U_j$$

of all the neighborhoods U_j. $U_X = U \cap V$ is an open neighborhood of the point X. Thus, for each point X there exists a neighborhood U_X open in D such that $N_h \cap U_X$ is either the empty set, or the union of a finite number of points and a finite number of pieces of analytic curves. By the Heine–Borel lemma, from the covering of the compact set K formed by all the neighborhoods U_X, one can extract a finite covering. The theorem is proved.

Denote by $N_h(a, b, c)$ the set of points of the domain D at which the equalities

$$\frac{\partial h}{\partial x} = a, \qquad \frac{\partial h}{\partial y} = b, \qquad \frac{\partial h}{\partial z} = c,$$

hold simultaneously, where h is a function harmonic in D. Let K be a compact subset of the domain D.

Corollary 1. The set $N_h(a, b, c) \cap K$ consists of a finite number of isolated points and a finite number of pieces of analytic curves.

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Received
15 IV 1964

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Note: Figure translations are in progress. See original paper for figures.