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Abstract

Full Text

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On Saturation Classes in the Theory of Singular Integrals

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1°. In my paper ([1]) the approximation of continuous 2π-periodic functions $f(x)$ by means of the singular integral

$$U_n[f;x] = \int_{-\pi}^{\pi} f(t)\Phi_n(t-x)\,dt, \tag{1}$$

was studied, where the continuous kernel $\Phi_n(t)$ was assumed to be a positive, even, 2π-periodic function for which

$$\int_{-\pi}^{\pi} \Phi_n(t)\,dt = 1, \qquad \int_0^{\pi} t\Phi_n(t)\,dt \xrightarrow[n\to\infty]{} 0. \tag{2}$$

In particular, it was shown that if * $\forall\sigma \in (0,\pi)$,

$$\int_{\sigma}^{\pi} \Phi_n(t)\,dt = o(\Delta_n), \qquad \left[\Delta_n = \int_0^{\pi} t^2\Phi_n(t)\,dt\right], \tag{3}$$

then whenever the finite derivative $f''(x)$ exists, one has

$$U_n[f;x] = f(x) + f''(x)\Delta_n + o(\Delta_n). \tag{4}$$

Following the terminology established in the theory of summation of Fourier series (see ([2])), we shall say that the approximation process by means of the integral (1) is **saturated with order of saturation** $\varphi(n)$, where $\varphi(n) > 0$ and $\varphi(n) \to 0$, if for every $f \in C_{2\pi}$ ($f \not\equiv$ const) one has

$$\max_x |\, U_n[f;x] - f(x)\,| > a(f)\varphi(n) \qquad (a(f) > 0,\ n = 1,2,3,...) \tag{5}$$

and at the same time there exists $f \in C_{2\pi}$ ($f \not\equiv$ const) for which

$$\max_x |\, U_n[f;x] - f(x)\,| < b(f)\varphi(n) \qquad (n = 1,2,3,...). \tag{6}$$

The set of functions $f \not\equiv$ const satisfying (6) is called the **saturation class** of the approximation process under consideration.

* In ([1]) it was noted that condition (3) is **sufficient** for the validity of the asymptotic formula (4). In fact, (3) is **necessary** for (4). Indeed, suppose that (4) is valid for some Δ_n. Putting $f(t) = t^2,\ x = 0$, we find

$$\int_0^\pi t^2 \Phi_n \, dt = \Delta_n + o(\Delta_n),$$

i.e. Δ_n necessarily has the indicated value (up to $o(\Delta_n)$). Next, taking $f(t) = t^4,\ x = 0$, we find

$$\int_\sigma^\pi t^4 \Phi_n \, dt = o(\Delta_n).$$

But for $\sigma \in (0, \pi)$ one has

$$0 \le \int_\sigma^\pi \Phi_n \, dt < \frac{1}{\sigma^4} \int_0^\pi t^4 \Phi_n \, dt.$$

2°. **Theorem 1.** Under conditions (2) and (3), the process of approximation by the integral (1) is saturated, with order of saturation Δ_n, and the saturation class is the set of functions $f(x)$ for which

$$f'(x) \in \mathrm{Lip}\, 1. \tag{7}$$

Proof. Construct the Fourier series of the function $\Phi_n(t)$:

$$\Phi_n(t) \sim \frac{1}{2\pi} + \sum_{k=1}^{\infty} \frac{1}{\pi} \rho_k^{(n)} \cos kt.$$

If the Fourier series for $f(t)$ is

$$A + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \tag{8}$$

then, by Parseval' s formula, we have

$$U_n[f; x] = A + \sum_{k=1}^{\infty} \rho_k^{(n)} (a_k \cos kx + b_k \sin kx). \tag{9}$$

Thus, approximation by the integral (1) is the summation of the series (8) with the aid of the numbers $\rho_k^{(n)}$. With regard to such summation, A. Kh. Turetskii

([3]) and F. I. Kharshiladze ([4]) independently established the following proposition: if $\varphi(n) > 0$, $\varphi(n) \to 0$, and for all k

$$\frac{1-\rho_k^{(n)}}{\varphi(n)} \underset{n\to\infty}{\longrightarrow} A_k \neq 0, \tag{10}$$

then the method $\{\rho_k^{(n)}\}$ is saturated with order of saturation $\varphi(n)$. If, in particular, $A_k = ak^2$, then the saturation class is the set of functions satisfying condition (7).

Keeping this result of Turetskii—Kharshiladze in mind, we must take $\varphi(n) = \Delta_n$ and consider the limit (10). It is clear that

$$\rho_k^{(n)} = \int_{-\pi}^{\pi} \Phi_n(t) \cos kt \, dt.$$

Hence,

$$1 - \rho_k^{(n)} = \int_{-\pi}^{\pi} (1 - \cos kt) \Phi_n(t) \, dt.$$

From this, according to (4), where $f(t) = 1 - \cos kt,\ x = 0$ has been put,

$$1 - \rho_k^{(n)} = k^2 \Delta_n + o(\Delta_n).$$

Consequently, for $k = 1, 2, 3, \ldots$ we shall have

$$\frac{1-\rho_k^{(n)}}{\Delta_n} \underset{n\to\infty}{\longrightarrow} k^2,$$

and our theorem is proved.

Remark 1. In the proof of the Turetskii—Kharshiladze theorem given in ([3]), it was additionally assumed that the kernel of the method under consideration is positive. It is clear that under the conditions of Theorem 1 this is so, since $\Phi_n(t) \geq 0$.

Remark 2. Theorem 1 is not applicable to the Fejér integral, since condition (3) is not fulfilled for it. However, the limit (10) exists here, with $A_k = k,\ \varphi(n) = 1/n$.

Remark 3. It is clear that for $k \geqslant 1$ one has $\rho_k^{(n)} \neq 1$. This condition is **necessary** for the validity of the theorem of Turetskii-Kharshiladze (although it is not stipulated by either of the authors).

3°. We now abandon condition (3), but instead assume that for some $\varphi(n) > 0$, $\varphi(n) \to 0$, one has

$$\lim_{n\to\infty} \frac{1-\rho_k^{(n)}}{\varphi(n)} = d_0 k^r + ... + d_r \quad (d_0 \neq 0). \tag{11}$$

This case was studied by A. Kh. Turetskii in ([2]), where it was proved that, for even r, the saturation class* of the method $\{\rho_k^{(n)}\}$ is the set of those f for which

$$f^{(r-1)}(x) \in \mathrm{Lip}\,1,$$

whereas for odd r it is the set of those f for which the conjugate function $\tilde{f}(x)$ has derivative

$$\tilde{f}^{(r-1)}(x) \in \mathrm{Lip}\,1.$$

A. Kh. Turetskii assumes here that the norms of the operators (9) are bounded.

Having noted this, let us consider the singular integral

$$Q_n[f;x] = \sum_{k=1}^{p+1} (-1)^{k-1} C_{p+1}^k U_n^{[k]}[f;x], \tag{12}$$

where

$$U_n^{[1]}[f;x] = U_n[f;x], \qquad U_n^{[k+1]}[f;x] = U_n[U_n^{[k]};x].$$

This integral was introduced by me in ([5]), where** it was shown that it possesses better approximation properties than the original integral (1). It is natural to pose the question of how to find the order and the saturation class of the integral (12), knowing this order and class for the original integral (1). In many cases this question can be answered immediately with the help of the theorem of A. Kh. Turetskii just cited.

Theorem 2. *If the integral (1) satisfies condition (11), then the process of approximation by the integral (12) is saturated with saturation order* $[\varphi(n)]^{p+1}$*. The saturation class is determined by the theorem of A. Kh. Turetskii depending on the parity of the number* $r(p+1)$*.*

Proof. If (1) is represented by formula (9), then

$$Q_n[f;x] = A + \sum_{k=1}^{\infty} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx),$$

where

$$\lambda_k^{(n)} = 1 - (1 - \rho_k^{(n)})^{p+1}.$$

Hence

$$\frac{1-\lambda_k^{(n)}}{[\varphi(n)]^{p+1}} = \left[\frac{1-\rho_k^{(n)}}{\varphi(n)}\right]^{p+1} \xrightarrow[n\to\infty]{} D_0 k^{r(p+1)} + \ldots + D_{r(p+1)},$$

where $D_0 \neq 0$. It remains to note that $\|U_n^{[k]}\| = 1$, whence

$$\|Q_n\| \leqslant \sum_{k=1}^{p+1} C_{p+1}^k \|U_n^{[k]}\| = 2^{p+1} - 1.$$

* The saturation order, according to what was said above, is $\varphi(n)$.

** The differentiability requirement on $\Phi_n(t)$, imposed in ([5]), is superfluous.

Example. If the original integral is the Fejér integral, then

$$Q_n[f;x] = A + \sum_{k=1}^{n} \left[1 - \left(\frac{k}{n}\right)^{p+1}\right] (a_k \cos kx + b_k \sin kx).$$

Thus, the known (see (4)) assertions on the orders and classes of saturation of the method of "typical means" follow from the properties of the Fejér integral and our Theorem 2.

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References

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[5] I. P. Natanson, DAN, **82**, No. 3, 337 (1952).

Note: Figure translations are in progress. See original paper for figures.