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# B. E. VEITS

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**Abstract**

**Full Text**

**B. E. VEITS**

**ON SOME STABILITY PROPERTIES OF BASES**

*(Presented by Academician V. I. Smirnov on 2 IV 1964)*

Let  $E$  be a separable Banach space over the field of complex numbers. By the method of Shauder <sup>(1)</sup>, the unit ball  $S^*$  ( $\|f\| \leq 1$ ) of the space  $E^*$  can be turned into a metric compactum in which convergence of a sequence with respect to the metric is equivalent to its weak convergence.

We shall call the series  $\sum_{k=1}^{\infty} y_k$  **regularly convergent** if the series  $\sum_{k=1}^{\infty} |f(y_k)|$  converges uniformly with respect to  $f \in S^*$  (such series were called strongly unconditionally convergent by I. M. Gel' fand <sup>(2)</sup>).

**Lemma.** The following four assertions are equivalent:

- a) the series  $\sum_{k=1}^{\infty} y_k$  converges unconditionally;
- b) the series  $\sum_{k=1}^{\infty} a_k y_k$  converges uniformly with respect to the choice of real numbers  $a_k, a_k = \pm 1, k = 1, 2, \dots$ ;
- c) the series  $\sum_{k=1}^{\infty} y_k$  converges regularly;
- d) the series  $\sum_{k=1}^{\infty} \varepsilon_k y_k$  converges uniformly with respect to the choice of complex numbers  $\varepsilon_k, |\varepsilon_k| \leq 1, k = 1, 2, \dots$

**Proof.** The implication a)  $\Rightarrow$  b) follows from Kadec' s lemma <sup>(3)</sup>.

b)  $\Rightarrow$  c). Let  $f \in S^*$ . Then for any  $x \in E, f(x) = f_1(x) + if_2(x)$ , where the functionals  $f_1$  and  $f_2$  are real, linear, and  $\|f_1\| \leq \|f\| \leq 1, \|f_2\| \leq \|f\| \leq 1$ .

Let  $|f_1(y_k)| = a_k f_1(y_k), |f_2(y_k)| = b_k f_2(y_k); a_k = \pm 1, b_k = \pm 1$ . Then for  $\varepsilon > 0$ , when  $n > m \geq N_\varepsilon$ ,

$$\left\| \sum_{k=m}^n a_k y_k \right\| < \frac{\varepsilon}{2}, \quad \left\| \sum_{k=m}^n b_k y_k \right\| < \frac{\varepsilon}{2},$$

$$\sum_{k=m}^n |f(y_k)| \leq \sum_{k=m}^n |f_1(y_k)| + \sum_{k=m}^n |f_2(y_k)| = f_1 \left( \sum_{k=m}^n a_k y_k \right) + f_2 \left( \sum_{k=m}^n b_k y_k \right) < \varepsilon.$$

c)  $\Rightarrow$  d). Let  $\varepsilon_k, |\varepsilon_k| \leq 1$ , be complex numbers. For natural  $n > m \geq N_\varepsilon$  there exists  $f_{mn} \in S^*$  such that

$$\left\| \sum_{k=m}^n \varepsilon_k y_k \right\| = f_{mn} \left( \sum_{k=m}^n \varepsilon_k y_k \right) = \sum_{k=m}^n \varepsilon_k f_{mn}(y_k) \leq \sum_{k=m}^n |f_{mn}(y_k)| < \varepsilon$$

d)  $\Rightarrow$  a). If  $(n_k)$  is an arbitrary increasing sequence of natural numbers, then, choosing  $\varepsilon_{n_k} = 1$  and  $\varepsilon_n = 0$  for  $n \neq n_k$ , we obtain the convergence of the series

$$\sum_{k=1}^{\infty} y_{n_k} = \sum_{n=1}^{\infty} \varepsilon_n y_n,$$

and this, by Orlicz' s theorem <sup>(4)</sup>, also means a).

A system  $(u_k)$  is called  $\omega$ -linearly independent if from the convergence of the series

$$\sum_{k=1}^{\infty} c_k u_k = \theta$$

there follow the equalities  $c_k = 0$ ,  $k = 1, 2, \dots$

In the papers <sup>(5,6)</sup> (see also <sup>(7)</sup>) the following criterion for the stability of bases was established:

**Theorem of M. G. Krein, M. A. Rutman, and D. P. Milman.** If  $(x_k)$  is a normalized basis of the space  $E$ ,  $(f_k)$  is the system of functionals biorthogonal to  $(x_k)$ , and an  $\omega$ -linearly independent system  $(u_k)$  satisfies the condition

$$\sum_{k=1}^{\infty} \|u_k - x_k\| \cdot \|f_k\| < +\infty, \quad (1)$$

then the system  $(u_k)$  is also a basis of the space  $E$ , equivalent to the basis  $(x_k)$ .

We shall prove a stronger criterion for the stability of bases.

**Theorem 1.** If  $(x_k)$  is a normalized basis of the space  $E$ , and an  $\omega$ -linearly independent system  $(u_k)$  satisfies the condition: the series

$$\sum_{k=1}^{\infty} (u_k - x_k) \quad (2)$$

converges unconditionally, then the system  $(u_k)$  is also a basis of the space  $E$ , equivalent to the basis  $(x_k)$ .

**Proof.** By virtue of the lemma, the unconditional convergence of the series (2) implies the convergence of the series

$$\sum_{k=1}^{\infty} |f(u_k - x_k)|, \quad (3)$$

uniformly with respect to  $f \in S^*$ . By a lemma of I. M. Gelfand <sup>(8)</sup>, from this it is easy to obtain the inequality

$$\sum_{k=1}^{\infty} |f(u_k - x_k)| \leq M \|f\| \leq M. \quad (4)$$

If  $(f_k)$ , as above, is the system of functionals biorthogonal to  $(x_k)$ :  $f_i(x_k) = \delta_{ik}$ ,  $i, k = 1, 2, \dots$ , then, in view of the normalizedness of the basis  $(x_k)$ ,  $\|f_k\| \leq a$ ,  $k = 1, 2, \dots$

For any  $m$  and  $n$  there is a functional  $\varphi_{mn} \in S^*$  such that

$$\begin{aligned} \left\| \sum_{k=m}^n f_k(x)(u_k - x_k) \right\| &= \sum_{k=m}^n f_k(x) \varphi_{mn}(u_k - x_k) \leq \\ &\leq \sum_{k=m}^n |f_k(x)| |\varphi_{mn}(u_k - x_k)| \leq \left( \sum_{k=m}^n \|f_k\| |\varphi_{mn}(u_k - x_k)| \right) \|x\| \leq \\ &\leq a \sum_{k=m}^n |\varphi_{mn}(u_k - x_k)| \|x\| \leq a\varepsilon \|x\|. \end{aligned}$$

Consequently, the uniformly convergent series

$$\sum_{k=1}^{\infty} f_k(x)(u_k - x_k) = Bx$$

defines a completely continuous operator  $B$ . Since the equality

$$Ax = (I + B)x = \sum_{k=1}^{\infty} f_k(x)u_k = \theta$$

is possible only (by virtue of the  $\omega$ -linear independence of the system  $(u_k)$ ) when  $f_k(x) = 0$ ,  $k = 1, 2, \dots$ , i.e., when  $x = \theta$ . Hence the operator  $A$  is continuously invertible and, consequently, the system  $(u_k)$ , where  $u_k = (I + B)x_k = Ax_k$ ,  $k = 1, 2, \dots$ , forms a basis in  $E$  equivalent to the basis  $(x_k)$ , as was required to prove.

**Remark.** By the theorem of Dvoretzky and Rogers <sup>(9)</sup>, in the space  $E$  there exists a series  $\sum_{k=1}^{\infty} y_k$  which converges unconditionally, but not absolutely:

$$\sum_{k=1}^{\infty} \|y_k\| = +\infty.$$

Put  $u_k = x_k - y_k$ . Then, if  $(x_k)$  is a normalized basis in the space  $E$ , then, by Theorem 1,  $(u_k)$  is a defective basis <sup>(10)</sup> in  $E$ , although the series  $\sum_{k=1}^{\infty} \|u_k - x_k\|$

diverges. Consequently, Theorem 1 is a strengthening of the theorem of M. G. Krein, M. A. Rutman, and D. P. Milman.

Following N. K. Bari <sup>(11)</sup>, we shall call a complete minimal system  $(x_k) \subset E$  a **Bessel system** if, for every  $x$ ,

$$\sum_{k=1}^{\infty} |f_k(x)|^2 < \infty,$$

and a **Hilbert system** if, for any sequence  $(\alpha_k) \in l^2$ , there exists, and moreover uniquely, an element  $x \in E$  for which  $f_k(x) = \alpha_k$ ,  $k = 1, 2, \dots$ , where  $f_i(x_k) = \delta_{ik}$ ,  $i, k = 1, 2, \dots$ . We shall call systems of elements  $(x_k)$  and  $(u_k)$  of the Banach space  $E$  **weakly quadratically close** if the series

$$\sum_{k=1}^{\infty} |f(u_k - x_k)|^2 \tag{5}$$

converges uniformly with respect to  $f \in S^*$ .

**Theorem 2.** *If  $(x_k)$  is a Bessel basis in the Banach space  $E$ , then every  $l^2$ -linearly independent system of elements  $(u_k) \subset E$ , weakly quadratically close to the basis  $(x_k)$ , is also a Bessel basis equivalent to the given basis  $(x_k)$ .*

**Corollary.** *If, in the space  $E$ , the Bessel bases  $(x_k)$  and  $(u_k)$  are weakly quadratically close, then their biorthogonal systems are also weakly quadratically close.*

Let now the space  $E = H$  be a separable Hilbert space. In the sequel, the main role is played by the following

**Lemma.** *In order that a linear operator  $T$  be completely continuous, it is necessary and sufficient that, for every orthonormal basis  $(e_k) \subset H$ , the series*

$$\sum_{k=1}^{\infty} |(Te_k, x)|^2 \tag{6}$$

converge uniformly with respect to  $x \in H$  and  $\|x\| \leq 1$ .

If  $A$  is a linear continuous operator in  $H$ , and  $(e_k)$  is some orthonormal basis in  $H$ , then in this basis the operator  $A$  is assigned the matrix  $(a_{ik})_1^{\infty}$ ,  $a_{ik} = (Ae_k, e_i)$ ,  $i, k = 1, 2, \dots$ , and since

$$|(Ae_k, x)|^2 = \left| \sum_{i=1}^{\infty} (x, e_i)(Ae_k, e_i) \right|^2 = \left| \sum_{i=1}^{\infty} a_{ik}(x, e_i) \right|^2,$$

\* The system  $(u_k)$  is called  $l^2$ -linearly independent if from the relations  $\sum c_k u_k = \theta$  and  $\sum |c_k|^2 < +\infty$  it follows that  $c_k = 0$ ,  $k = 1, 2, \dots$

then a necessary and sufficient condition for the complete continuity of the operator  $A$  is the convergence, uniformly with respect to  $x \in H$  and  $\|x\| \leq 1$ , of the series

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ik}(x, e_i) \right|^2. \quad (7)$$

Condition (7) may be called the **weak quadratability** of the matrix  $(a_{ik})_1^{\infty}$ .

For brevity, let us call a basis weakly quadratically close to some orthonormal basis a  $(W)$ -basis. Obviously,  $(W)$ -bases are Riesz bases <sup>(11)</sup>.

**Theorems 3, 4, and 5** establish some properties and internal criteria for  $(W)$ -bases.

**Theorem 3.** If  $(x_k)$  is a  $(W)$ -basis, then the biorthogonal system to it also forms a  $(W)$ -basis, weakly quadratically close to the same orthonormal basis as the given basis  $(x_k)$ .

**Theorem 4.** In order that a Bessel basis  $(x_k)$  be a  $(W)$ -basis, it is necessary and sufficient that the biorthogonal systems  $(x_k)$  and  $(y_k)$ ,  $(x_i, y_k) = \delta_{ik}$ ,  $i = 1, 2, \dots$ , be weakly quadratically close.

**Corollary.** In order that a Bessel basis  $(x_k)$  be a  $(W)$ -basis, it is necessary and sufficient that the operator  $I - C$  be completely continuous, where  $C$  is the operator carrying the biorthogonal system  $(y_k)$  into the basis  $(x_k)$ :

$$x_k = C y_k, \quad k = 1, 2, \dots$$

**Theorem 5** (cf. <sup>(12)</sup>). In order that a complete system  $(x_k)$  in  $H$  be a  $(W)$ -basis, it is necessary and sufficient that: 1) the elements  $(x_k)$ ,  $k = 1, 2, \dots$ , be  $l^2$ -linearly independent; 2) the matrix  $((x_i, x_k) - \delta_{ik})_1^{\infty}$  be weakly quadratable.

Murmansk Pedagogical Institute

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*Note: Figure translations are in progress. See original paper for figures.*

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