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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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COMPLEX PLANE DEFORMATION

1. Initial assumptions

Let a slowly varying homogeneous state of stress be given, determined by known functions of time $\sigma_x(t)$, $\sigma_y(t)$, $\tau_{xy}(t)$, and by the conditions $\tau_{xz} = \tau_{yz} = 0$. We shall call a plane plastic deformation one for which the component of plastic deformation in the direction of the Oz axis is equal to zero. Under the Huber-Mises plasticity condition this leads [1] to the fact that $\sigma_z = \frac{1}{2}(\sigma_x + \sigma_y)$, while under the Tresca condition plane plastic deformation is possible when σ_z is the mean principal stress, i.e.,

$$\left|\sigma_z - \frac{1}{2}(\sigma_x - \sigma_y)\right| < \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}. \tag{1,1}$$

We note that for $\sigma_z = \frac{1}{2}(\sigma_x + \sigma_y)$ the constancy of the octahedral shear stress means the constancy of the maximum shear stress. The difference in the indicated plasticity conditions is reflected only in the restrictions imposed on the value of σ_z, which is of no significance for the problems considered in the present paper.

Consider the influence of an infinitesimal slip $d\Gamma_n$ in the direction n (Fig. 1a), orthogonal to the axis Oz. This slip causes a change in the resistance to plastic shear dS_m, different in different directions m. We shall assume that the ratio $dS_m/d\Gamma_n$ depends only on the angle between the directions m and n.

Fig. 1

In accordance with the law of parity of shear stresses, simultaneously with slip in the direction n there must occur slip in the direction l, perpendicular to n (Fig. 1b). The total increment of resistance to shear may be represented in the form

$$dS_m = F(\omega)\, d\gamma_{nl}, \tag{1,2}$$

where $d\gamma_{nl}$ denotes the total angle of shear due to slips in the directions n and l ($d\gamma_{nl} = 2d\Gamma_n$); ω is the angle determined by the directions m and n.

If the body is cut by mutually orthogonal planes along which the indicated slips occur simultaneously, then, during motion in the directions n and l, the cut-off parts of the body will be on different sides (right and left). We shall assign the index n to that direction in which the part of the body cut off by the corresponding plane, when moving, will be on the same side (either right or left) as when moving in the direction m. As the argument of the function F in formula (1,2), we choose the angle ω, $0 \leq \omega \leq \pi/2$, determined by the directions m and n, as shown in Fig. 1b. We then set

$$F(\omega) = k \ln(c/\omega), \tag{1,3}$$

where k and c are constants. In what follows, materials are considered for which the function F, characterizing sensitivity to deformation anisotropy, is determined by formula (1,3), and their mechanical properties are elucidated.

2. Limits of applicability of deformation theory

Let the maximum shear stress T be mono-

tonically increasing function of time t. At a certain instant t_0 it reaches the initial resistance to shear τ_s; at this, slip occurs along the directions of action of the maximum tangential stress. In the indicated directions the maximum hardening will occur, as a result of which subsequent slips will take place along a set of mutually perpendicular directions. Slips in directions enclosed between the angles θ and $\theta + d\theta$ (Fig. 2) give a plastic shear strain, which we shall represent in the form $d\Gamma_n = \frac{1}{2}\varphi(\theta, t)d\theta$, where $\varphi(\theta, t)$ is the shear intensity, subject to determination.

Fig. 2

By monotonic plastic deformation we shall mean one for which the slip intensity increases with time, i.e., when

$$\partial\varphi(\theta, t)/\partial t > 0 \tag{2,1}$$

for all directions θ in which slips have occurred. We shall show that monotonic deformation is possible when the directions of the principal stresses rotate, with a certain restriction on the rate of this rotation.

Choose the coordinate axes so that, at the instant of occurrence of plastic deformation, they coincide with the principal directions and so that at this instant the condition $\sigma_y > \sigma_x$ is satisfied. Let $\Phi(t)$ denote the angle of rotation of the principal directions. Then the tangential stress in the direction m, determined by the angle θ_0 (Fig. 2), is represented in the form

$$\tau_m(\theta_0, t) = T(t) \cos 2[\theta_0 - \Phi(t)], \quad \text{where} \tag{2,2}$$

$$T(t) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}, \qquad \Phi(t) = \frac{1}{2}\operatorname{arc\,tg}\frac{2\tau_{xy}}{\sigma_x - \sigma_y}.$$

In the directions of developing slip, the tangential stress must be equal to the resistance to shear, which for the time being we shall regard as independent of the normal stress. Hence it follows that

$$\tau_s + \int_{-\alpha_1(t)}^{\alpha_2(t)} \varphi(\theta, t) \ln \frac{c}{|\theta - \theta_0|}\, d\theta = T(t) \cos 2[\theta_0 - \Phi(t)], \tag{2,3}$$

where $\alpha_{1,2}(t)$ are new unknown functions determining the set of directions along which slips have occurred. Assuming $|\theta_0 - \Phi(t)| \ll 1$, we obtain

$$k \int_{-\alpha_1(t)}^{\alpha_2(t)} \varphi(\theta, t) \ln \frac{c}{|\theta - \theta_0|}\, d\theta = T(t)\{1 - 2[\theta_0 - \Phi(t)]^2\}. \tag{2,4}$$

A continuous solution of this integral equation will be

$$\varphi(\theta, t) = \frac{4T(t)}{k\pi}\sqrt{[\alpha_1(t) + \theta][\alpha_2(t) - \theta]}, \tag{2,5}$$

where $\alpha_{1,2}(t)$ is determined by the formulas

$$\alpha_{1,2}(t) = \alpha(t) \mp \Phi(t), \tag{2,6}$$

$$\alpha^2(t)\left[\frac{1}{2} + \ln 2c - \ln \alpha(t)\right] = \frac{1}{2} - \frac{\tau_s}{2T(t)}. \tag{2,7}$$

A graphical representation of the function α is given in Fig. 3.

Condition (2,1) gives a restriction on the rate of rotation of the principal directions under which monotonic deformation occurs:

$$|\Phi'(t)| < \alpha'(t). \tag{2,8}$$

For the limiting rate ($|\Phi'(t)| = \alpha'(t) - 0$) we obtain $\Phi(t) = \pm\alpha(t)$.

The total plastic deformation from all slips is determined by the components

$$e_x = -e_y = -\frac{1}{2}\int_{-\alpha_1(t)}^{\alpha_2(t)} \varphi(\theta, t) \cos 2\theta\, d\theta, \qquad e_{xy} = -\int_{-\alpha_1(t)}^{\alpha_2(t)} \varphi(\theta, t) \sin 2\theta\, d\theta.$$

Fig. 3

Figure 2: Fig. 3

Fig. 4

Figure 3: Fig. 4

Substituting here $\cos 2\theta \simeq 1 - 2\theta^2$, $\sin 2\theta \simeq 2\theta$, and formulas (2.6) and (2.7), we obtain

$$e_x = -e_y = -\frac{2T(t)}{k}\alpha^2(t)\left[1 - \frac{\alpha^2(t)}{2} - 2\Phi^2(t)\right],$$

$$e_{xy} = +\frac{8}{k}T(t)\Phi(t)\alpha^2(t). \tag{2.9}$$

Consequently, the plastic deformation is completely determined by the stress tensor $(\sigma_x, \sigma_y, \tau_{xy})$ at the given instant of time, if the maximum shear stress is a monotonically increasing function of time, and the rate of rotation of its direction is bounded by condition (2.8). The diagram of the dependence $T \sim e_y$, constructed on the basis of formulas (2.7) and (2.9) for $\Phi'(t) = 0$, $c = \pi/2$, is shown in Fig. 4.

Fig. 3

Fig. 4

3. **The Bauschinger effect.** Consider the case when, after a proportional increase of the loads beyond the elastic range, their direction of action changes to the opposite. Determine the shear resistance S_{-m} in the direction opposite to m (Fig. 2). In this case $\omega = \frac{1}{2}\pi - (\theta - \theta_0)$, and, consequently,

$$S_{-m}(\theta_0, t) = \tau_s + \frac{4T(t)}{\pi}\int_{-\alpha_1(t)}^{\alpha_2(t)} \sqrt{\alpha^2(t) - \theta^2}\,\ln\frac{2c}{\pi - 2|\theta - \theta_0|}\,d\theta.$$

Assuming $\alpha \ll 1$, $|\theta_0| \leq \alpha$, and using formula (2.7), we find

$$S_{-m}(\theta_0, t) = T(t)\left\{1 - \alpha^2(1 + 2\ln\pi/\alpha) - \frac{8}{\pi^2}\left[\theta_0^2\sqrt{\alpha^2 - \theta_0^2} + \frac{2}{3}\sqrt{(\alpha^2 - \theta_0^2)^3}\right]\right\}, \tag{3.1}$$

where we have neglected small quantities of fourth order. Taking formula (2.9) into account, it follows from the last result that, with the adopted degree of accuracy,

$$S_{-m}(0,t) = T(t) - \frac{1}{2}ke_y(t)\left[1 + 2\ln\pi/\alpha(t) + \frac{16\alpha(t)}{3\pi^2}\right]$$

or

$$S_{-m}(0,t) = S_m(0,t) - ke_y(t)\left[\ln(5.17/\alpha(t)) + \frac{16\alpha(t)}{3\pi^2}\right]. \tag{3.2}$$

The last term shows by how much the shear resistance has decreased when the direction of the load is changed to the opposite.

4. **Arbitrary continuous plastic deformation.** Suppose that at some instant of time t slips occur in directions determined by the angle θ (Fig. 2), and we shall assume that $\beta_1(t) \leq \theta \leq \beta_2(t)$.

In the indicated slip directions, the increment of the shear resistance during the time dt is equal to the increment of the shear stresses. It follows from this that

$$k\int_{\beta_1(t)}^{\beta_2(t)} \varphi_t'(\theta,t)\ln\frac{c}{|\theta-\theta_0|}\,d\theta = \frac{\partial}{\partial t}\tau_m(\theta_0,t). \tag{4.1}$$

Using formula (2,2), we have

$$\partial\tau_m(\theta_0,t)/\partial t = T'(t)\cos 2[\theta_0 - \Phi(t)] + 2T(t)\Phi'(t)\sin 2[\theta_0 - \Phi(t)]. \tag{4,2}$$

In view of the fact that the slip directions are close to the direction of the maximum shear stress, we shall assume that $\beta_2(t) - \beta_1(t) \ll 1$, $|\theta_0 - \Phi(t)| \ll 1$. Therefore one may write

$$\partial\tau_m(\theta_0,t)/\partial t = T_0(t) + T_1(t)\theta_0 + T_2(t)\theta_0^2, \tag{4,3}$$

where

$$\begin{aligned} &T_0(t) = T'(t) - 4T(t)\Phi(t)\Phi'(t) - 2T'(t)\Phi^2(t), \\ &T_1(t) = 4\,[T'(t)\Phi(t) + T(t)\Phi'(t)], \qquad T_2(t) = -2T'(t). \end{aligned}$$

For given limits of integration (β_1, β_2), the solution of the integral equation (4,1) will be

$$\varphi_t'(\theta,t) = -\frac{2T_2}{k\pi}\sqrt{(\beta_2-\theta)(\theta-\beta_1)} + \frac{[T_1 + (\beta_1+\beta_2)T_2]\theta + f_0}{k\pi\sqrt{(\beta_2-\theta)(\theta-\beta_1)}}, \tag{4,4}$$

where

$$\begin{aligned} f_0 = {} & [\ln(\beta_2-\beta_1)/c]^{-1}[T_0 + \tfrac{1}{2}(\beta_2+\beta_1)T_1 + \tfrac{1}{4}(\beta_2+\beta_1)^2T_2 + \tfrac{1}{8}(\beta_2-\beta_1)^2T_2] \\ & + [\tfrac{1}{2}(\beta_1+\beta_2)T_1 + \tfrac{1}{2}(\beta_1+\beta_2)^2T_2 - \tfrac{1}{4}(\beta_2-\beta_1)^2T_2]. \end{aligned}$$

Further, one can find the rate of change of the shear resistance for any θ_0

$$S'_t[\theta_0, t, \beta_1(t), \beta_2(t)] = k\int_{\beta_1(t)}^{\beta_2(t)} \varphi'_t(\theta, t)\ln\frac{c}{\omega}\,d\theta. \tag{4,5}$$

With the accepted accuracy, from the last formula we obtain:

$$S'_t = \begin{cases} T_0 + T_1\theta_0 + T_2\theta_0^2, & \text{for } \beta_1 \le \theta_0 \le \beta_2, \\ \psi_1(\theta_0, t, \beta_1, \beta_2), & \text{for } \beta_2 - \pi/2 \le \theta_0 \le \beta_1,\ \beta_2 \le \theta_0 \le \pi/2 - \beta_1, \\ \psi_2(\theta_0, t, \beta_1, \beta_2), & \text{for } \beta_1 + \pi/2 \le \theta_0 \le \beta_2 + \pi/2, \end{cases} \tag{4,6}$$

$$S'_t[\pi + \theta_0, t, \beta_1, \beta_2] = S'_t(\theta_0, t, \beta_1, \beta_2),$$

where

$$\begin{aligned} \psi_1 = {} & T_0 + T_1\theta_0 + T_2\theta_0^2 - \tfrac{1}{2}[2T_1 + T_2(\beta_1+\beta_2+2\theta_0)]\sqrt{(\theta_0-\beta_2)(\theta_0-\beta_1)} \\ & - [f_0 + \tfrac{1}{2}(\beta_1+\beta_2)T_1 + \tfrac{1}{2}(\beta_1+\beta_2)^2T_2 - \tfrac{1}{4}(\beta_2-\beta_1)^2T_2]\times \\ & \quad \times \ln\left|\frac{2\sqrt{(\theta_0-\beta_2)(\theta_0-\beta_1)} + 2\theta_0 - \beta_1 - \beta_2}{\beta_2-\beta_1}\right|; \end{aligned}$$

$$\begin{aligned} \psi_2 = {} & \tfrac{1}{4}\,[4f_0 + 2(\beta_1+\beta_2)T_1 + 2(\beta_1+\beta_2)^2T_2 - (\beta_2-\beta_1)^2T_2]\times \\ & \times \ln(2c/\pi) + (1/4\pi^2)\,[2(\beta_1+\beta_2)^2 + (\beta_2-\beta_1)^2 - 4(\pi-2\theta_0)(\beta_1+\beta_2) \\ & \qquad + 2(\pi-2\theta_0)^2]\,f_0 + (1/2\pi^2)\,[8f_0 + 2(\pi-2\theta_0)T_1 + 3(\beta_1+\beta_2)T_1]\times \\ & \times \sqrt{(\beta_2-\theta_0+\pi/2)(\theta_0-\beta_1-\pi/2)} + (1/4\pi^2)\{4[2f_0 + (\pi-2\theta_0)T_1]\times \\ & \times (\beta_1+\beta_2) + 8f_0(\pi-2\theta_0) + 2(\beta_1+\beta_2)^2T_1 + (\beta_2-\beta_1)^2T_1\}\times \\ & \times \arcsin\left[\frac{\pi - 2\theta_0 + \beta_1 + \beta_2}{\beta_2-\beta_1}\right]. \end{aligned}$$

Over the set Ω of directions defined by the inequality $\beta_1(t) \le \theta \le \beta_2(t)$, along which slips are occurring at the given instant of time t, the shear resistance S_m is equal to the shear stress $\tau_m(\theta, t)$; outside this region $S_m > \tau_m(\theta, t)$. It follows that

$$\tau_s + \int_{t_0}^{t} S'_t[\theta, t, \beta_1(t), \beta_2(t)]\,dt = \begin{cases} T(t)\cos 2[\theta - \Phi(t)], & \theta \subset \Omega, \\ T(t)\cos 2[\theta - \Phi(t)] + \eta, & \theta \not\subset \Omega,\ \eta > 0. \end{cases}$$

The functions $\beta_1(t)$, $\beta_2(t)$ must be determined from this condition. Their explicit determination is possible only in particular cases.

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CITED LITERATURE

1. Yu. N. Rabotnov, *Strength of Materials*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.

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