M. Rosenblatt-Roth

1964

SovietRxiv

Abstract

Full Text

M. Rosenblatt-Roth

ON THE RELATIVE STABILITY OF SUMS OF NON-NEGATIVE RANDOM VARIABLES LINKED IN A MARKOV CHAIN

(Presented by Academician A. N. Kolmogorov, 11 I 1964)

Let ξ_k $(k = 1, 2, \ldots)$ be a sequence of random variables linked in a Markov chain and capable of taking only nonnegative values; let a_i be the coefficient of ergodicity ([1,2]) of the i-th transition probability function of the chain; let

$$\alpha^{(n)} = \min_{1 \le i < n} \alpha_i > 0, \qquad S_n = \sum_{i=1}^{n} \xi_i.$$

We shall call the sums S_n **relatively stable** ([3,4]) if there exists some sequence of numbers $A_n > 0$ $(n = 1, 2, \ldots)$ such that

$$\mathbf{P}\left(\left|A_n^{-1} S_n - 1\right| \ge \delta\right) \to 0 \qquad (n \to \infty) \tag{1}$$

for every $\delta > 0$.

Theorem 1. In order that the sums S_n of nonnegative random variables linked in a Markov chain be relatively stable, it is sufficient that there exist some sequence of numbers $A_n > 0$ $(n = 1, 2, \ldots)$ such that, for every $\varepsilon > 0$,

$$\sum_{k=1}^{n} \int_{|x - a_k| \ge \varepsilon A_n \alpha^{(n)}} dF_k(x) \to 0 \qquad (n \to \infty), \tag{2}$$

$$\frac{1}{A_n} \sum_{k=1}^{n} \int_{|x - a_k| < \varepsilon A_n \alpha^{(n)}} x \, dF_k(x) \to 1 \qquad (n \to \infty), \tag{3}$$

where $F_k(x)$ is the unconditional distribution function of the random variable ξ_k, and $a_k = A_k - A_{k-1}$. In this case the numbers A_n may be taken as normalizing divisors.

Theorem 2. The relative stability of sums S_n of nonnegative random variables linked in a Markov chain with normalizing divisors $A_n \to A$ $(n \to \infty)$ entails the following:

a) if $A = \infty$, the summands ξ_k $(k = 1, 2, \ldots, n)$ are asymptotically constant, i.e., there exist certain numbers b_k $(k = 1, 2, \ldots, n)$ such that, however small $\delta > 0$ may be,

$$\mathbf{P}\left(A_n^{-1} |\xi_k - b_k| \ge \delta\right) \to 0 \qquad (n \to \infty) \tag{4}$$

uniformly with respect to $k \le n$;

b) if $A < \infty$, the summands ξ_k $(k = 1, 2, ...)$ are, with probability 1, equal to certain numbers a'_k.

Theorem 3. In order that the sums S_n of nonnegative random variables linked in a Markov chain have relative stability with a prescribed normalizing divisor $A_n \to \infty$ $(n \to 0)$, it is sufficient that, for every $\varepsilon > 0$, conditions (2) and (3) hold, where $a_k = A_k - A_{k-1}$. For $A_n \to A < \infty$ this is false.

Theorem 4. In order that the sums S_n of nonnegative random variables ξ_k $(k = 1, 2, ...)$, linked in a Markov chain and possessing finite-

with finite mathematical expectations $\mathbf{M}\xi_k$, are relatively stable with normalizing divisor $A_n = \mathbf{M}S_n$, it is sufficient that for every $\varepsilon > 0$ conditions (2) and (3) be fulfilled, where $a_k = \mathbf{M}\xi_k$. In this case the possibility $A_n \to A < +\infty$ is not excluded.

Theorem 5. In order that the sums S_n of nonnegative random variables ξ_k $(k = 1, 2, ...)$, linked into a Markov chain, which possess relative stability with normalizing divisor A_n, consist of summands ξ_k for which $A_n^{-1}\xi_k$ $(k = 1, 2, \dots, n)$ are negligible in the limit, i.e., however small $\delta > 0$ may be,

$$\mathbf{P}\big(A_n^{-1}\xi_k \geq \varepsilon\big) \to 0 \quad (n \to \infty) \tag{5}$$

uniformly with respect to $k \leq n$, it is necessary and sufficient that

$$A_n \to +\infty, \qquad A_n^{-1}A_{n-1} \to 1 \quad (n \to \infty). \tag{6}$$

Theorem 6. In order that the sums S_n of nonnegative random variables ξ_k $(k = 1, 2, ...)$, linked into a Markov chain, be relatively stable and that the quantities $B_n^{-1}\xi_k$ $(k = 1, 2, \dots, n)$, where B_k is a normalizing divisor, be negligible in the limit, it is sufficient that there exist numbers A_n $(n = 1, 2, ...)$ for which the conditions

$$\sum_{k=1}^{n} \mathbf{P}\big(\xi_k \geq \varepsilon A_n \alpha^{(n)}\big) \to 0, \qquad (n \to \infty); \tag{7}$$

$$\frac{1}{A_n}\sum_{k=1}^{n}\int_0^{\varepsilon A_n \alpha^{(n)}} x\, dF_k(x) \to 1 \qquad (n \to \infty), \tag{8}$$

hold, however $\varepsilon > 0$ may be chosen. Here the numbers A_n may be taken as the normalizing divisors B_n.

Theorem 7. In order that the sums S_n of nonnegative random variables ξ_k $(k = 1, 2, ...)$, linked into a Markov chain, be relatively stable with a prescribed normalizing divisor A_n, and that the quantities $A_n^{-1}\xi_k$ $(k = 1, 2, \dots, n)$ be negligible in the limit, it is sufficient that, for any $\varepsilon > 0$, relations (7) and (8) be fulfilled.

Theorem 8. In order that the sums S_n of nonnegative random variables ξ_k $(k = 1, 2, \dots)$, linked into a Markov chain, possessing finite mathematical expectations $\mathbf{M}\xi_k$, be relatively stable with normalizing divisor $A_n = \mathbf{M}S_n$, and that the quantities $A_n^{-1}\xi_k$ $(k = 1, 2, \dots, n)$ be negligible in the limit, with $\alpha^{(n)} > \rho > 0$ $(n = 1, 2, \dots)$, it is sufficient that, for every $\varepsilon > 0$, the relation

$$\frac{1}{A_n}\sum_{k=1}^{n}\int_0^{\varepsilon A_n \alpha^{(n)}} x\,dF_k(x) \to 1 \quad (n \to \infty). \tag{9}$$

be fulfilled.

Theorem 9. In order that the sums S_n, linked into a stationary and homogeneous Markov chain with ergodicity coefficient $\alpha > 0$, of nonnegative identically distributed random variables be relatively stable, it is sufficient that there exist some numbers $A_n > 0$ $(n = 1, 2, \dots)$ satisfying the conditions

$$n\int_{\varepsilon A_n \alpha^{(n)}}^{+\infty} dF(x) \to 0 \qquad (n \to \infty); \tag{10}$$

$$\frac{n}{A_n}\int_0^{\varepsilon A_n \alpha^{(n)}} x\,dF(x) \to 1 \qquad (n \to \infty). \tag{11}$$

for any $\varepsilon > 0$, where $F(x)$ is the unconditional distribution function of the random variables. In this case the numbers A_n may be taken as normalizing divisors.

Remark. If the random variables ξ_k $(k = 1, 2, \dots)$ are independent, i.e. $\alpha^{(n)} = 1$ $(n = 1, 2, \dots)$, then from these theorems, as special cases, follow the corresponding results of [3,4].

Faculty of Mathematics and Mechanics
University of Bucharest
Bucharest, Romanian People' s Republic

Received
13 V 1963

References

[1] R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, no. 1, 72 (1956).
[2] R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, no. 4, 365

(1956).
[3] A. Khintchine, *Giornale dell' Istituto Italiano degli Attuari*, **7**, 365 (1936).
[4] A. A. Bobrov, *Scientific Notes of Moscow State University*, issue 146, Mathematics, **3**, 92 (1950).

Note: Figure translations are in progress. See original paper for figures.