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## Abstract

## Full Text

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## On Some Criteria for the Existence of Stable Periodic Solutions of Quasilinear Parabolic Equations

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**1.** Let $\Omega$ be a bounded open domain in $n$-dimensional space whose boundary $\Gamma$ is sufficiently smooth. Consider the parabolic equation

$$\frac{\partial u}{\partial t} = Lu + f(t, x, u). \tag{1}$$

Here $L$ is a second-order elliptic operator:

$$Lu = \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} - a(x)u, \tag{2}$$

where

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geq r_0 \sum_{k=1}^{n} \xi_k^2 \quad (r_0 > 0),$$

$$a_{ik}(x) = a_{ki}(x), \qquad a(x) \geq 0.$$

The coefficients of the differential expression (2) are assumed to be sufficiently smooth. The function $f(t, x, u) = f(t, x_1, \dots, x_n, u)$ is also assumed to be sufficiently smooth.

We shall assume that $f(t, x, u)$ has the property of $\omega$-periodicity in $t$: $f(t + \omega, x, u) \equiv f(t, x, u)$, and we shall consider the question of the existence for equation (1) of solutions $u(t, x)$ that are $\omega$-periodic in $t$ $(-\infty < t < \infty,\ x \in \overline{\Omega},\ \overline{\Omega} = \Omega + \Gamma)$, satisfying the boundary condition

$$u(t, x) = 0 \quad (x \in \Gamma). \tag{3}$$

To study this problem we shall use the general method proposed by M. A. Krasnosel' skii [1]. It consists in writing equation (1), together with the boundary condition (3), in the form of an operator differential equation

$$\frac{du}{dt} = Lu + f(t, u) \tag{4}$$

in a suitably chosen function space $E$ (in the theorems proved below, equation (4) is considered in the space $C_0$ of functions continuous on $\overline{\Omega}$ and vanishing on $\Gamma$). To each initial condition $u(0) = u_0$ of equation (4) there corresponds the value $Tu_0$ of the solution of equation (4) at $t = \omega$. If the operator $T$ leaves invariant some cone in the space $E$, then general principles of the existence of fixed points for positive operators are applicable to proving the existence of periodic solutions. If, moreover, the operator $T$ turns out to be concave, then the corresponding periodic solutions are stable in the sense of Lyapunov. If, on the other hand, the operator $T$ is convex, then the corresponding periodic solutions are unstable.

The scheme described above was applied to the study of systems of ordinary differential equations in a paper by the author and M. A. Krasnosel' skii—

skii' s paper ([2]) and in the paper of M. A. Krasnosel' skii ([3]). Some applications to problems on periodic solutions of equations with partial derivatives are indicated in the report of M. A. Krasnosel' skii and P. E. Sobolevskii at the Soviet-American symposium of 1963 in Novosibirsk. Let us also note that some existence theorems for periodic solutions of equations (1) were obtained by D. Kh. Karimov ([4]), J. Prodi ([5,6]), and I. I. Shmulev ([7]) by other methods.

The transition from problem (1)–(3) to equation (4) and the investigation of the operator $T$ can be carried out either by classical methods or by methods using ideas from the theory of semigroups ([8–10]).

2. All further constructions are carried out under the assumption that

$$f(t, x, 0) \geq 0 \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega}). \tag{5}$$

We denote by $\lambda_0$ the least eigenvalue of the operator $Au = -Lu$ $(u(x) = 0,\ x \in \Gamma)$ ($\lambda_0 > 0$, since this operator has (see (1)) a positive inverse).

**Theorem 1.** Suppose that the inequality

$$f(t, x, u) \leq au + a_1 \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega},\ u \geq 0), \tag{6}$$

is satisfied, where $a < \lambda_0$. Then problem (1)–(3) has at least one nonnegative $\omega$-periodic solution.

3. In the remaining part of the paper we shall assume that problem (1)–(3) has the zero solution:

$$f(t, x, 0) \equiv 0 \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega}). \tag{7}$$

**Theorem 2.** Suppose that the conditions of Theorem 1 are satisfied and that

$$f(t, x, u) \geq bu \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega},\ 0 \leq u \leq \rho_0), \tag{8}$$

where $b > \lambda_0$. Then problem (1)–(3) has at least one $\omega$-periodic nonnegative solution distinct from the identically zero solution.

**Theorem 3.** Suppose that the inequalities

$$f(t, x, u) > \alpha u - \alpha_1 (0 \leq t \leq \omega,\ x \in \overline{\Omega},\ u \geq 0), \tag{9}$$

$$f(t, x, u) \leq \beta u (0 \leq t \leq \omega,\ x \in \overline{\Omega},\ 0 \leq u \leq \rho_0), \tag{10}$$

are satisfied, where $\beta < \lambda_0 < \alpha$. Then problem (1)–(3) has at least one nonnegative $\omega$-periodic solution distinct from the identically zero solution.

4. We shall say that the function $f(t, x, u)$ is strongly concave if

$$u f'_u(t, x, u) - f(t, x, u) \leq 0 \qquad (0 \leq t \leq \omega,\ x \in \overline{\Omega},\ u \geq 0), \tag{11}$$

and the left-hand side is strictly negative for some $t = t_0$ and all $x \in \overline{\Omega}$, $u > 0$. It turns out that the strong concavity of the function $f(t, x, u)$ implies the $u_0$-concavity (see ([1])) of the operator $T$ on the cone $K \subset C_0$ of functions nonnegative on $\Omega$.

**Theorem 4.** Suppose that $f(t, x, u)$ is strongly concave. Suppose that the conditions of one of Theorems 1 or 2 are satisfied. Then problem (1)–(3) has one and only one nonnegative $\omega$-periodic solution distinct from the identically zero solution.

5. We shall say that a nonnegative $\omega$-periodic solution $u_0(t, x)$ of problem (1)–(3) is stable if every solution $u(t, x)$ satisfying a nonzero and nonnegative initial condition has the property that

$$\lim_{t \to \infty} \|u(t, x) - u_0(t, x)\| = 0. \tag{12}$$

**Theorem 5.** Suppose that the conditions of Theorem 4 are satisfied. Then the nonnegative $\omega$-periodic solution of problem (1)–(3), distinct from the identically zero solution, is stable.

**Theorem 6.** Suppose that the conditions of Theorem 3 are satisfied and that $f(t, x, u)$ is strongly convex in the sense that

$$u f'_u(t, x, u) - f(t, x, u) \geqslant 0 \qquad (0 \leqslant t \leqslant \omega,\ x \in \overline{\Omega},\ u \geqslant 0) \tag{13}$$

and the left-hand side, for some $t = t_0$ and all $x \in \overline{\Omega}$, $u > 0$, is strictly positive. Then every nonnegative and non-identically-zero $\omega$-periodic solution of problem (1)–(3) is unstable.

6. We note here one general formula that was used in proving the positivity of the operator $T$ and that, it seems to us, is of independent interest. Let $u(x)$ $(x \in \overline{\Omega})$ be a continuous function, and let $\Omega_+$ be the set of points at which $u(y) = \|u\| = \max_{y\in\Omega} |u(y)|$, and $\Omega_-$ the set of points at which $u(y) = -\|u\|$. The following equality holds:

$$\lim_{\tau\to+0} \frac{1}{\tau}[\|u(x) + \tau h(x)\| - \|u(x)\|] = \max\left\{\max_{x\in\overline{\Omega}+} h(x),\ -\min_{x\in\overline{\Omega}-} h(x)\right\}. \qquad (14)$$

This formula was pointed out by M. A. Krasnosel' skii.

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*Note: Figure translations are in progress. See original paper for figures.*