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Abstract

Full Text

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APPLICATION OF GENERALIZED CHARACTERISTIC NUMBERS TO THE STUDY OF THE STABILITY OF AN EQUILIBRIUM POINT

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§ 1. Definition of generalized characteristic numbers.

In an n-dimensional real Euclidean space E_0 consider a domain Ξ containing the origin O_0. Put

$$\dot{\Xi} \stackrel{\text{def}}{=} \Xi \setminus O_0.$$

Denote the boundary of Ξ by Ξ_{gr} (Ξ_{gr} may also turn out to be the empty set). Assign a continuous function

$$x : (-\infty, +\infty) \to \dot{\Xi}$$

to the class $\mathfrak{R}$ if the α-limit set of x coincides with O_0, while the ω-limit set is contained in Ξ_{gr}. Thus,

$$\lim_{\tau \to -\infty} x(\tau) = O_0, \qquad \operatorname{Lim}_{\tau \to +\infty} x(\tau) \subset \Xi_{\text{gr}}.$$

On the domain $\dot{\Xi}$ define a scalar function v, filling out by its values $(0, +\infty)$, i.e.

$$v : \dot{\Xi} \xrightarrow{\text{onto}} (0, +\infty).$$

Denote by Γ_γ the level set of the function v corresponding to the number γ, i.e.

$$\Gamma_\gamma \stackrel{\text{def}}{=} \{\xi \mid \xi \in \dot{\Xi},\ v(\xi) = \gamma\}.$$

For $x \in \mathfrak{R}$ and $\gamma > 0$ put

$$\underline{\tau}(x, \gamma) \stackrel{\text{def}}{=} \inf\{\tau \mid x(\tau) \in \Gamma_\gamma\}, \qquad \overline{\tau}(x, \gamma) \stackrel{\text{def}}{=} \sup\{\tau \mid x(\tau) \in \Gamma_\gamma\}.$$

Suppose that the function v has the following properties: v_1) each Γ_γ, $\gamma > 0$, is compact; v_2) for any x, $x \in \mathfrak{R}$, and all $\gamma, \tilde{\gamma}$ $(0 < \gamma < \tilde{\gamma})$ one has

$$\underline{\tau}(x, \gamma) < \underline{\tau}(x, \tilde{\gamma}), \qquad \overline{\tau}(x, \gamma) < \overline{\tau}(x, \tilde{\gamma}).$$

We note that all sets Γ_γ, $\gamma > 0$, are nonempty and all quantities $\underline{\tau}(x,\gamma)$, $\overline{\tau}(x,\gamma)$ are finite.

Next consider a function $d(\gamma_1,\gamma_2)$, defined for all positive values of the arguments and assuming all possible real values:

$$d : (0,+\infty) \times (0,+\infty) \xrightarrow{\text{onto}} (-\infty,+\infty).$$

Suppose that for all $\gamma_1,\gamma_2,\gamma_3,\gamma$ $(0 < \gamma_1 < \gamma_2 < \gamma_3,\ 0 < \gamma)$ the following hold:

$$\mathrm{d}_1)\ d(\gamma,\gamma) = 0; \qquad \mathrm{d}_2)\ 0 < d(\gamma_2,\gamma_1) = -d(\gamma_1,\gamma_2);$$

$$\mathrm{d}_3)\ d(\gamma_2,\gamma) > d(\gamma_1,\gamma); \qquad \mathrm{d}_4)\ d(\gamma_3,\gamma_2) + d(\gamma_2,\gamma_1) \geq d(\gamma_3,\gamma_1).$$

Take any function

$$x : (-\infty,+\infty) \to \dot{\Xi}.$$

Put

$$D(x,\tau,\tau_0) \stackrel{\text{def}}{=} d\{v[x(\tau_0+\tau)],\ v[x(\tau_0)]\},$$

where $\tau_0 \in (-\infty,+\infty)$, $\tau \in (0,+\infty)$.

Associate with the function x the generalized characteristic numbers: the **lower** vd**-number**

$$\overline{\Omega}\, vd\, x \stackrel{\text{def}}{=} \max\left\{ \overline{\lim_{\tau\to+\infty}} \frac{1}{\tau} D(x,\tau,\tau_0),\ -\underline{\lim_{\tau\to+\infty}} \frac{1}{\tau} D(x,-\tau,\tau_0) \right\}$$

and the vd**-number**

$$\Omega^* vd\, x \stackrel{\text{def}}{=} \overline{\lim_{\tau\to+\infty}} \frac{1}{\tau} \sup_{-\infty<\tau_0<+\infty} D(x,\tau,\tau_0).$$

We note that from $\mathrm{d}_{1-4})$ it follows that

$$\sup_{\gamma>0} |d(\gamma_2,\gamma) - d(\gamma_1,\gamma)| \leq 2|d(\gamma_2,\gamma_1)|,$$

and therefore $\overline{\Omega}\, vd\, x$ does not depend on τ_0. The lower vd-number is either finite ...

number, or by an improper number $-\infty$, or by an improper number $+\infty$, i.e. $\overline{\Omega}\, vd\, x \in [-\infty,+\infty]$. Similarly $\overset{*}{\Omega}\, vd\, x \in [-\infty,+\infty]$. As is not hard to see,

$$\overline{\Omega}\, vd\, x \leq \overset{*}{\Omega}\, vd\, x.$$

§ 2. Structure of a neighborhood of an unstable equilibrium point.

Consider the differential system

$$dx/dt = f(x), \qquad x \in \Xi, \tag{1}$$

with right-hand side $f(x)$ satisfying a local Lipschitz condition on Ξ and vanishing at O_0. The origin O_0 is therefore an equilibrium point. Suppose that all solutions of (1) are continuable in both directions, i.e. are defined for all values of the argument τ. The solution $x(\tau)$ with initial value $x(0) = \xi$ will be denoted by $x(\tau, \xi)$. The set of all α-limit points of solutions of (1) distinct from the trivial one will be denoted by A.

Assume that O_0 is unstable (here and below instability is understood in the sense of Lyapunov). If $\nu = 2$ and O_0 is an isolated equilibrium point, then there exists a solution $x(\tau, \xi_0)$, $\xi_0 \in \Xi$, such that $x(\tau, \xi_0) \to O_0$ as $\tau \to -\infty$, i.e. $O_0 \in A$. Examples show that for some systems of type (1), when $\nu > 2$, an isolated unstable equilibrium point may fail to be an α-limit point for any solution $x(\tau, \xi)$, $\xi \in \Xi$, and, consequently, $O_0 \notin A$. However, the following theorem is valid:

Theorem. *Let O_0 be unstable. Then $O_0 \in \overline{A}$.*

Outline of the proof. Suppose, to the contrary, that $O_0 \notin \overline{A}$. Denote by ε_0 a number such that the entire closed sphere $S(\varepsilon_0)$ of radius ε_0 with center at O_0 is contained in Ξ and has no common points with A. Take an arbitrary ε from the interval $(0, \varepsilon_0)$. To each point σ of the surface $S_{\text{rp}}(\varepsilon)$ of the sphere $S(\varepsilon)$ assign the number $\tau(\sigma)$—the greatest of the nonpositive moments of strict entry of $x(\tau, \sigma)$ into $S(\varepsilon)$, i.e. $x(\tau, \sigma) \in S(\varepsilon)$ for all $\tau \in [\tau(\sigma), 0]$ and $x(\tau_n, \sigma) \notin S(\varepsilon)$, where $\{\tau_n\}$ is some increasing sequence converging to $\tau(\sigma)$. By virtue of the continuous dependence of solutions of (1) on initial values, the function $\tau(\sigma)$ is lower semicontinuous, i.e.

$$\lim_{\sigma \to \sigma_0} \tau(\sigma) \geq \tau(\sigma_0).$$

Consequently, $\tau(\sigma)$ is bounded below on $S_{\text{rp}}(\varepsilon)$ by some number τ^*. The vector function $f(x)$ is continuous on $S(\varepsilon)$ and vanishes at O_0; therefore there is a positive $\delta = \delta(\varepsilon)$ such that every solution of (1) requires a time, greater in absolute value than $|\tau^*|$, to pass from $S_{\text{rp}}(\delta)$ to $S_{\text{rp}}(\varepsilon)$. Consequently, none of the solutions $x(\tau, \xi)$, $\xi \in S(\delta)$, intersects the surface $S_{\text{rp}}(\varepsilon)$ for $\tau > 0$, i.e. the point O_0 turns out to be stable, contrary to the hypothesis of the theorem.

§ 3. Criteria for stability and instability of an equilibrium point.

Necessary criterion for instability. *If O_0 is unstable, then in every neighborhood u of this point there are initial values of nontrivial solutions (1) with nonnegative small vd-numbers.*

Proof. On the basis of the theorem of § 2 there exists a solution $x(\tau, \xi_0)$, $\xi_0 \in \Xi$, and points $a \in u$ such that $x(\tau_n, \xi_0) \to a$ for some sequence $\{\tau_n\} \to -\infty$. Without loss of generality, $\xi_0 \in u$. The sequence $\{x(\tau_n, \xi_0)\}$ is contained inside some surface Γ_{γ_0}; therefore there exists a number Δ such that for all n

$$d\{v[x(0, \xi_0)],\, v[x(\tau_n, \xi_0)]\} > \Delta,$$

$$\frac{1}{\tau_n}\, d\{v[x(\tau_n, \xi_0)],\, v[x(0, \xi_0)]\} > \frac{\Delta}{-\tau_n},$$

$$\tau_n^* \overset{\text{def}}{=} -\tau_n, \qquad -\lim_{n\to\infty} \frac{1}{\tau_n^*}\, d\{v[x(-\tau_n, \xi_0)],\, v[x(0, \xi_0)]\} \geq 0,$$

i.e.

$$\overline{\Omega}\, vd\, [x(\tau, \xi_0)] \geq 0.$$

Sufficient criterion for instability. If in any neighborhood of the point O_0 there are initial values of solutions of (1) with positive vd-numbers, then O_0 is unstable.

Proof follows from the fact that $\overset{*}{\Omega}\ \mathrm{vd}[x(\tau, \xi)] > 0$ entails the existence of a sequence $\{\tau_n\} \to +\infty$ such that either $\{x(-\tau_n, \xi)\} \to O_0$, or $\{x(\tau_n, \xi)\} \to \Xi_{\mathrm{gr}}$.

Sufficient criterion for asymptotic stability. If the small vd-numbers of all nontrivial solutions (1) with initial values from a sufficiently small neighborhood of the point O_0 are negative, then O_0 is asymptotically stable.

Proof. The stability of O_0 follows from the necessary criterion for instability. Moreover, from $\overline{\Omega}\mathrm{vd}[x(\tau, \xi)] < 0$ it follows that $x(\tau, \xi) \to O_0$ as $\tau \to +\infty$, and hence O_0 is a point of attraction for every solution $x(\tau, \xi)$, and therefore O_0 is asymptotically stable.

Necessary criterion for asymptotic stability. If O_0 is asymptotically stable, then there exist functions v_0 and d_0, satisfying the conditions of §1 and such that the $v_0 d_0$-numbers of all nontrivial solutions (1) with sufficiently small initial values are negative.

Proof. Without loss of generality, we take Ξ to be the domain of attraction of O_0. From the asymptotic stability of O_0 it follows that there exists a Lyapunov function $v_0(x)$, satisfying conditions v_{1-2} and strictly decreasing along every nontrivial solution of (1) (see, for example, ([1]), pp. 214-218). Take arbitrary γ_1 and γ_2, $0 < \gamma_1 < \gamma_2$. For any point

$$\xi \in \Gamma_{\gamma_1\gamma_2} \stackrel{\mathrm{def}}{=} \bigcup_{\gamma_1<\gamma<\gamma_2} \Gamma_\gamma$$

define the number $T(\xi, \gamma_2, \gamma_1)$—the time during which $x(\tau, \xi)$ remains strictly inside $\Gamma_{\gamma_1\gamma_2}$. For given γ_1 and γ_2 the set $\{T(\xi, \gamma_2, \gamma_1)\}$ is bounded above. Put

$$d_0(\gamma_2, \gamma_1) \stackrel{\mathrm{def}}{=} \sup\{T(\xi, \gamma_2, \gamma_1)\}, \qquad d(\gamma_1, \gamma_2) \stackrel{\mathrm{def}}{=} -d(\gamma_2, \gamma_1), \qquad d(\gamma_1, \gamma_1) \stackrel{\mathrm{def}}{=} 0,$$

where the supremum is taken over all $\xi \in \Gamma_{\gamma_1\gamma_2}$. The function $d_0(\gamma_2, \gamma_1)$ satisfies conditions d_{1-4}. From the definition of $d_0(\gamma_2, \gamma_1)$ it follows that

$$\sup_{-\infty<\tau_0<+\infty} d_0\{v_0[x(\tau_0+\tau)],\, v_0[x(\tau_0)]\} \le -\tau$$

and

$$\overset{*}{\Omega}\, v_0 d_0[x(\tau, \xi)] \le -1$$

for any $\xi \in \Xi$.

§ 4. **Criterion for conditional stability.** Denote by m a certain closed subset of Ξ. Theorem §2 can be strengthened in the direction that, if O_0 is unstable with respect to perturbations from m, then $O_0 \in \bar{A} \cap m$, and hence in m there exist nontrivial solutions (1) with arbitrarily small (in norm) initial values and nonnegative small vd-numbers (necessary criterion for instability). Hence there follows a sufficient criterion for conditional asymptotic stability.

If the small vd-numbers of all nontrivial solutions (1) with initial values from m are negative, then O_0 is asymptotically stable with respect to perturbations from m.

§ 5. **Generalized exponents.** The vd-number and the small vd-number are a natural generalization of the Lyapunov characteristic exponent (with sign changed)

$$\bar{\omega}x \stackrel{\mathrm{def}}{=} \lim_{\tau\to+\infty} \frac{1}{\tau} \ln\{\|x(\tau_0+\tau)\|/\|x(\tau_0)\|\}.$$

The results of §3 show that the method of generalized characteristic numbers is universal in detecting the asymptotic stability of an equilibrium point in the same sense in which Lyapunov' s second method is universal. A direct generalization—the generalized characteristic exponent of Lyapunov—ob ···

the generalized exponent $\overline{\Omega}x$ and the small generalized exponent $\overset{*}{\Omega}x$ can be obtained if one sets $v(x) = \|x\|$, $d(\gamma_1, \gamma_2) \equiv \ln \frac{\gamma_1}{\gamma_2}$, i.e.

$$\overline{\Omega}x \stackrel{\mathrm{def}}{=} \max\left\{ \overline{\lim_{\tau\to+\infty}} \frac{1}{\tau} \ln \frac{\|x(\tau_0+\tau)\|}{\|x(\tau_0)\|},\ -\underline{\lim_{\tau\to+\infty}} \frac{1}{\tau} \ln \frac{\|x(\tau_0+\tau)\|}{\|x(\tau_0)\|} \right\},$$

$$\overset{*}{\Omega}x \overset{\text{def}}{=} \overline{\lim_{\tau\to+\infty}} \frac{1}{\tau} \sup_{-\infty<\tau_0<+\infty} \ln \frac{\|x(\tau_0+\tau)\|}{\|x(\tau_0)\|}.$$

If x is a solution of the **linear** system (1), then $\overline{\Omega}x = \overset{*}{\Omega}x = \overline{\omega}x$. In contrast to $\overline{\omega}x$ ([2,3]), the generalized characteristic number Ωx, and still more $\overset{*}{\Omega}x$, has the property that the negativity of the generalized characteristic numbers of all nontrivial solutions of (1) with sufficiently small initial values in norm ensures the asymptotic stability of the rest point O_0 (see § 3). It is not difficult to show that the relation

$$\overline{\Omega}x = \max\{\overline{\omega}x, -\underline{\alpha}x\},$$

where $\underline{\alpha}x$ is the lower exponent of the function $x(-\tau)$, i.e.

$$\underline{\alpha}x \overset{\text{def}}{=} \underline{\lim_{\tau\to+\infty}} \frac{1}{\tau} \ln \frac{\|x(\tau_0-\tau)\|}{\|x(\tau_0)\|},$$

or, in other words, the minus-exponent of the solution x ([4]), always holds. For weakly nonlinear systems the criteria of § 3 for the specific values of v and d indicated in the present paragraph are consequences of known results ([4]).

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Note: Figure translations are in progress. See original paper for figures.