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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON THE DESCRIPTION OF EXTENSIONS OF POSITIVE DEFINITE KERNELS

*(Presented by Academician N. N. Bogolyubov, 8 V 1964)*

**1°.** In this note a theorem is established on the representation of positive definite (p.d.) kernels whose values are operators in a Hilbert space  $H$ , in terms of eigenfunctions of ordinary differential equations (in the finite-dimensional case of  $H$  such a theorem was established by M. G. Krein <sup>(1,7)</sup>). Then, using this result, all extensions of p.d. kernels satisfying certain conditions from a strip or rectangle to the whole space are described. Our description generalizes to general kernels the description obtained in the case of ordinary p.d. functions by means of the methods of the theory of functions by B. Ya. Levin and I. E. Ovcharenko <sup>(5)</sup>. The method of proof makes essential use of Yu. M. Berezanskiĭ's theorems on the differentiation of an operator measure <sup>(2)</sup>.

**2°.** Let  $H$  be a Hilbert space with scalar product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$ . We shall denote elements of this space by  $f, g, \dots$ . Consider in  $H$  a strongly continuous operator function  $K(s, t)$  of the point  $(s, t)$  ( $s, t \in (a, b)$ ,  $-\infty \leq a$ ,  $b \leq \infty$ ). The function  $K(s, t)$  is called a p.d. kernel if, for arbitrary  $f_1, \dots, f_n \in H$  and  $t_1, \dots, t_n \in (a, b)$  ( $n = 1, 2, \dots$ ),

$$\sum_{i,j=1}^n (K(t_i, t_j) f_i, f_j) \geq 0. \quad (1)$$

From (1) it is easy to show that

$$(K(t, t)f, f) \geq 0, \quad K^*(s, t) = K(t, s),$$

$$|(K(s, t)f, g)|^2 \leq (K(s, s)f, f)(K(t, t)g, g) \quad (f, g \in H).$$

Denote by  $C_0^\infty(H, (a, b))$  the space of all infinitely differentiable (in the strong sense) finite vector-functions defined on  $(a, b)$  with values in  $H$ . Introduce on  $C_0^\infty(H, (a, b))$  the bilinear form

$$\langle u, v \rangle_K = \int_a^b \int_a^b (K(s, t)u(t), v(s)) ds dt,$$

which satisfies all the requirements of a scalar product; however, generally speaking, from  $\langle u, u \rangle_K = 0$  it does not follow that  $u = 0$ . The totality of those  $u$  for which  $\langle u, u \rangle_K = 0$  forms a linear set. Taking the quotient space by this set and then completing it, we obtain a certain Hilbert space  $\mathcal{L}_K(H, (a, b))$ .

Consider on  $(a, b)$  the ordinary differential expression

$$L(u) = \sum_{0 \leq k \leq r} a_k(s) \frac{d^k u}{ds^k} \quad (2)$$

with complex-valued coefficients  $a_k(s)$ , each of which is  $k + 1$  times continuously differentiable inside  $(a, b)$ . Denote by  $\chi_j(s; \lambda)$  the fundamental system of solutions of the equation  $Lu = \lambda u$ , satisfying the initial conditions

$$d^k \chi_j(s; \lambda) / ds^k \Big|_{s=\xi} = \delta_{jk} \quad (j, k = 0, \dots, r-1), \quad (3)$$

where  $\xi$  is some point of  $(a, b)$ .

**Theorem 1.** Let  $K(s, t)$  ( $a \leq s, t \leq b$ ) be a p.d. strongly continuous operator kernel, for which inside  $(a, b)$  there exist weak derivatives  $\partial^{2k} K(s, t) / \partial s^k \partial t^k$  ( $k = 0, \dots, r-1$ ), and let  $L$  be a differential expression of the form (2) with  $k + 1$  times continuously differentiable coefficients  $a_k(s)$ . In order that the representation be valid in the weak sense,

$$K(s, t) = \int_{-\infty}^{\infty} \sum_{j,k=0}^{r-1} \chi_j(s; \lambda) \overline{\chi_k(t; \lambda)} d\tau_{jk}(\lambda), \quad (4)$$

where  $\|d\tau_{jk}(\lambda)\|_{j,k=0}^{r-1}$  ( $-\infty < \lambda < \infty$ ) is some finite nonnegative matrix function (i.e. the function  $\sum_{i,j=1}^n (\tau_{ij}(\lambda) f_i, f_j)$  is nondecreasing for all  $f_i \in H$ ), composed of bounded operators  $\tau_{jk}(\lambda)$  in the space  $H$ , normalized by the condition  $\|\tau_{jk}(-\infty)\| = 0$  and continuous from the left in the strong sense, it is necessary and sufficient that

$$\langle L^* u, v \rangle_K = \langle u, L^* v \rangle_K, \quad u, v \in C_0^\infty(H, (a, b)) \quad (5)$$

( $L^*$  is the differential expression formally adjoint to  $L$ ). The matrix  $\|d\tau_{jk}(\lambda)\|_{j,k=0}^{r-1}$  is determined uniquely by the kernel  $K(s, t)$  if and only if the closure in  $\mathcal{L}_K(H, (a, b))$  of the operator  $u \rightarrow L^* u$ ,  $u \in C_0^\infty(H, (a, b))$ , is maximal.

In the case when  $H$  is finite-dimensional, Theorem 1 can easily be obtained by using the method of guiding functionals <sup>(1)</sup> or with the help of an expansion in generalized eigenfunctions of self-adjoint operators <sup>(2,3)</sup>. If  $H$  is infinite-dimensional, then the representation (4) is obtained by a limiting transition

from the finite-dimensional case, and then the uniqueness condition for the matrix  $\|d\tau_{jk}(\lambda)\|_{j,k=0}^{r-1}$  is established independently. We note that, as in the finite-dimensional case <sup>(1)</sup>, the matrix  $\|d\tau_{jk}(\lambda)\|_{j,k=0}^{r-1}$  and the self-adjoint extensions of the operator  $u \rightarrow L^*u$ ,  $u \in C_0^\infty(H, (a, b))$ , are in one-to-one correspondence.

We observe that Theorem 1 is also valid when the kernel  $K(s, t)$  is only continuous. Then, in the representation (4), the matrix function  $\|d\tau_{jk}(\lambda)\|_{j,k=0}^{r-1}$  is not, in general, finite on the whole axis.

3°. Let  $G = (a_1, b_1) \times (a_2, b_2)$ , where  $-\infty \leq a_i, b_i \leq \infty$  ( $i = 1, 2$ ). Suppose that in  $G \times G$  an ordinary p.d. kernel  $K(x, y)$  ( $x, y \in G$ ) is given, which is a continuous function of the point  $(x, y) \in G \times G$ . Consider differential expressions

$$L^{(i)}u(t) = \sum_{\alpha=0}^{r_i} a_\alpha^{(i)}(t) \frac{d^\alpha u(t)}{dt^\alpha} \quad (t \in (-\infty, \infty); i = 1, 2) \quad (6)$$

of type (2). Suppose that  $L^{(i)}$  ( $i = 1, 2$ ) \*-commute with the kernel  $K(x, y)$  in the corresponding variables, i.e., in the sense of distributions of L. Schwartz,

$$L_{x_i}^{(i)}K(x_1, x_2; y_1, y_2) = \overline{L_{y_i}^{(i)}}K(x_1, x_2; y_1, y_2) \quad (i = 1, 2) \quad (x_i, y_i \in (a_i, b_i)) \quad (7)$$

(the bar denotes passage to complex-conjugate coefficients).

Let now  $\varphi(x_1)$  be an arbitrary function from  $C_0^\infty(a_1, b_1)$ . Then the kernel

$$K_\varphi(x_2, y_2) = \int_{a_1}^{b_1} \int_{a_1}^{b_1} K(x_1, x_2; y_1, y_2) \varphi(y_1) \overline{\varphi(x_1)} dx_1 dy_1 \quad (x_2, y_2 \in (a_2, b_2))$$

is p.d. and \*-commutes with  $L^{(2)}$ . Therefore

$$K_\varphi(x_2, y_2) = \int_{-\infty}^{\infty} \sum_{i,j=0}^{r_2-1} \overline{\chi_j(y_2; \lambda)} \chi_i(x_2; \lambda) d\zeta_{ij}^{(\varphi)}(\lambda), \quad (8)$$

where  $\chi_j(x_2, \lambda)|_{j=0}^{r_2-1}$  is a fundamental system of solutions of the equation  $L^{(2)}u = \lambda u$  of type (3), and  $\|d\sigma_{ij}^{(\varphi)}(\lambda)\|_{i,j=0}^{r_2-1}$  is a nonnegative matrix, normalized in the usual way.

**Theorem 2.** Let  $K(x, y)$  ( $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in G$ ) be a p.d. kernel for which the derivatives  $\partial^{2k}K(x_1, x_2; y_1, y_2)/\partial x_2^k \partial y_2^k$  ( $k = 0, 1, \dots, r_2 - 1$ ) exist, and let  $L^{(i)}$  be differential expressions of the form (6), \*-commuting with  $K(x, y)$  in the corresponding variables, i.e., the equalities (7) are satisfied. Suppose

that for arbitrary  $\varphi \in C_0^\infty(a_1, b_1)$ ,  $K_\varphi(x_2, y_2)$  admits a unique representation (8), which means that the matrix  $\|d\sigma_{ij}^{(\varphi)}(\lambda)\|_{i,j=0}^{r_2-1}$  is uniquely determined by the kernel  $K_\varphi(x_2, y_2)$ . Then

$$K(x_1, x_2; y_1, y_2) = \int_{-\infty}^{\infty} \sum_{i,j=0}^{r_2-1} \chi_i(x_2; \lambda) \overline{\chi_j(y_2; \lambda)} \psi_{ij}(\lambda; x_1, y_1) d\sigma(\lambda), \quad (9)$$

where  $d\sigma(\lambda)$  ( $-\infty < \lambda < \infty$ ) is a finite measure, and  $\psi(\lambda, x_1, y_1) = \|\psi_{ij}(\lambda; x_1, y_1)\|_{i,j=0}^{r_2-1}$  is a matrix-function which: a) for every fixed  $\lambda$  is p.d. and \*-commutes with  $L^{(1)}$ ; b) for every fixed  $(x_1, y_1)$  is  $\sigma$ -integrable with respect to  $\lambda$ . If the measure  $d\sigma(\lambda)$  is fixed in the representation (9), then the matrix-function  $\psi_{ij}(\lambda; x_1, y_1)$  is determined uniquely by the kernel  $K(x, y)$ . Conversely, every representation of the form (9) defines a certain p.d. kernel.

Let us indicate the course of the proof. For simplicity we shall assume that  $-\infty < a_1, b_1 < \infty$ . Considering the integral representation of the form (4) of the p.d. operator kernel

$$[K(x_2, y_2)\varphi](x_1) = \int_{a_1}^{b_1} K(x_1, x_2; y_1, y_2)\varphi(y_1) dy_1$$

in the space  $\mathcal{L}_2(a_1, b_1)$ , we arrive at the fact that the operator  $\tau(\infty) = \sum_{i=0}^{r_2-1} \tau_{i,i}(\infty)$  has finite trace

$$\text{Sp } \tau(\infty) = \int_{a_1}^{b_1} \sum_{i=0}^{r_2-1} \frac{\partial^{2i} K(x_1, x_2; x_1, y_2)}{\partial x_2^i \partial y_2^i} dx_1 \Big|_{x_2=y_2=\xi} < \infty,$$

and therefore  $\sigma(\lambda) = \text{Sp } \tau(\lambda) < \text{Sp } \tau(\infty)$  for every  $\lambda$ . Denote  $\psi_{ij}(\lambda) = d\tau_{ij}(\lambda)/d\sigma(\lambda)$ . Substituting now the value  $d\tau_{jk}(\lambda)$  into equality (4), written in the case of our kernel in the weak sense for a  $\delta$ -shaped sequence, by a limiting passage we obtain the required representation.

It follows from Theorem 2 that, for fixed measure  $d\sigma(\lambda)$  in the representation (9), every extension of the p.d. kernel  $K(x_1, x_2; y_1, y_2)$  from the domain  $G \times G$ ,  $G = (a_1, b_1) \times (a_2, b_2)$ ,  $-\infty < a_1, b_1 < \infty$ ,  $-\infty \leq a_2, b_2 \leq \infty$ , to a larger domain  $G' \times G'$ ,  $G' = (a'_1, b'_1) \times (a_2, b_2)$ ,  $(a'_1, b'_1) \supset (a_1, b_1)$ , preserving p.d. and \*-commutativity with  $L^{(1)}$ , gives rise to an extension of the matrix  $\|\psi_{jk}(\lambda; x_1, y_1)\|_{j,k=0}^{r_2-1}$  having properties a), b) on  $(a'_1, b'_1) \times (a_1, b_1)$ .

Conversely, every extension of the matrix  $\|\psi_{jk}(\lambda; x_1, y_1)\|_{j,k=0}^{r_2-1}$  from  $(a_1, b_1) \times (a_1, b_1)$  to  $(a'_1, b'_1) \times (a'_1, b'_1)$  with properties a), b) on  $(a'_1, b'_1) \times (a'_1, b'_1)$ , by

formula (9), defines an extension of the kernel  $K(x, y)$  from  $G \times G$  to  $G' \times G'$ , satisfying the conditions of Theorem 2.

4°. Let us give several examples.

A. Let a continuous function of two variables  $f(x_1, x_2)$  ( $-2l < x_1 < 2l < \infty$ ,  $-\infty < x_2 < \infty$ ) be such that the kernel  $K(x_1, x_2; y_1, y_2) = f(x_1 - y_1, x_2 - y_2)$  ( $-l < x_1, y_1 < l$ ,  $-\infty < x_2, y_2 < \infty$ ) is p.d. The differential expressions  $L^{(1)} = L^{(2)} = i d/dt$  \*-commute with  $K(x_1, x_2; y_1, y_2)$ . Using Bochner's theorem (4), it is not difficult to verify that all the conditions of Theorem 2 are satisfied for the given kernel. Therefore

$$f(x_1, x_2) = \int_{-\infty}^{\infty} e^{i\lambda x_2} \psi_{00}(\lambda, x_1) d\sigma(\lambda),$$

where  $d\sigma(\lambda)$  is a certain positive measure, and  $\psi_{00}(\lambda, x_1)$  is a p.d. function of  $x_1$  for each fixed  $\lambda$ , and is  $\sigma$ -summable with respect to  $\lambda$  for each fixed  $x_1 \in (-2l, 2l)$ .

Choosing  $\sigma(\lambda)$  so that  $\psi_{00}(\lambda, 0) = 1$ , we obtain a description of all extensions of a p.d. function of two variables from the strip  $(-2l, 2l) \times (-\infty, \infty)$  to the whole plane, which was given by B. Ya. Levin and I. E. Ovcharenko (5), namely: *the problem of describing all extensions of a p.d. function given in the strip  $-2l \leq x_1 \leq 2l$ ,  $-\infty < x_2 < \infty$ , to a wider strip  $-2l' \leq x_1 \leq 2l'$ ,  $-\infty < x_2 < \infty$ ,  $l' > l$ , reduces to the question of extending the p.d. function  $\psi_{00}(\lambda, x_1)$  of one variable  $x_1$  from the interval  $(-2l, 2l)$  to the interval  $(-2l', 2l')$  for every fixed  $\lambda$ , with preservation of positive definiteness; moreover these extensions must be  $\sigma$ -summable with respect to  $\lambda$  (even  $\sigma$ -measurability is sufficient).* A description of extensions of p.d. functions of one variable was given by M. G. Krein (6).

B. Let the continuous function  $f(x_1, x_2) = f(-x_1, x_2)$  ( $-2l < x_1 < 2l < \infty$ ,  $-\infty < x_2 < \infty$ ) be such that the kernel

$$K(x_1, x_2; y_1, y_2) = \frac{1}{2} [f(x_1 + y_1, x_2 - y_2) + f(x_1 - y_1, x_2 - y_2)]$$

is p.d. In this case  $L^{(1)} = d^2/dt^2$ ,  $L^{(2)} = i d/dt$ . Using Theorem 2 and the evenness of  $f(x_1, x_2)$  in  $x_1$ , we obtain that

$$f(x_1, x_2) = \int_{-\infty}^{\infty} e^{i\lambda x_2} \psi_{00}(\lambda, x_1) d\sigma(\lambda),$$

where  $d\sigma(\lambda)$  is a positive measure chosen so that  $\psi_{00}(\lambda, 0) = 1$ ;  $\psi_{00}(\lambda, x_1)$ , for fixed  $\lambda$ , is an even p.d. function, i.e. such that the kernel

$$\frac{1}{2} [\psi_{00}(\lambda, x_1 + y_1) + \psi_{00}(\lambda, x_1 - y_1)]$$

is p.d., and for fixed  $x$  is  $\sigma$ -summable with respect to  $\lambda$ .

Therefore, *the description of extensions of  $f(x_1, x_2)$  to a wider strip reduces to finding all even p.d. extensions of the function  $\psi_{00}(\lambda, x_1)$* , which was also done by M. G. Krein.

C. Let the continuous function  $f(x_1, x_2)$  ( $-2l_1 < x_1 < 2l_1 < \infty$ ,  $-2l_2 < x_2 < 2l_2 \leq \infty$ ) be such that the kernel  $K(x_1, x_2; y_1, y_2) = f(x_1 - y_1, x_2 + y_2)$  ( $-l_i < x_i < l_i$ ,  $i = 1, 2$ ) is p.d. Here  $L^{(1)} = i d/dt$ ,  $L^{(2)} = d/dt$ . Using the theorem of S. N. Bernstein <sup>(4)</sup>, it is not hard to verify that all the conditions of Theorem 2 are satisfied. Therefore

$$f(x_1, x_2) = \int_{-\infty}^{\infty} e^{\lambda x_2} \psi_{00}(\lambda, x_1) d\sigma(\lambda),$$

where  $d\sigma(\lambda)$  and  $\psi_{00}(\lambda, x_1)$  are the same as in example A.

Thus, *the description of extensions of  $f(x_1, x_2)$  to a wider rectangle reduces to the description of all extensions of  $\psi_{00}(\lambda; x_1)$* , as in example A.

Other analogous examples can also be constructed.

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