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Abstract

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ON SOME INEQUALITIES FOR FUNCTIONS FROM WEIGHTED CLASSES AND BOUNDARY-VALUE PROBLEMS WITH STRONG DEGENERATION ON THE BOUNDARY

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1. In this note we give Poincaré-type inequalities for functions whose derivatives are p-summable over a domain g with certain weights. The significance of these inequalities may be seen from applications to the theory of boundary-value problems for elliptic equations with degeneration on the boundary Γ of the domain g.

For degenerating equations, as was first shown by M. V. Keldysh ([1]), the formulation of boundary-value problems depends essentially on the character of the degeneration; in particular, that part of the boundary where "strong" degeneration occurs may be freed from boundary conditions. Subsequent studies by M. I. Vishik, S. G. Mikhlin, L. D. Kudryavtsev, and others (see the survey ([2])) developed and generalized these results (see also the works ([3–8])).

In the present paper we consider an analogue of the first boundary-value problem for a degenerating elliptic equation of order $2r$, $r \geqslant 1$. The degeneration is characterized by the behavior of the corresponding form when approaching the boundary (see (10)). In the simplest case this form vanishes as some power β of the distance to the boundary—we then speak of homogeneous degeneration (along the whole boundary).

In the works cited, boundary-value problems with "strong" degeneration were considered for second-order equations, and in this case it was assumed that strong degeneration occurs not on the whole boundary but on a part of it. If, however, boundary values were prescribed on the entire boundary Γ of the domain g, then the number s of boundary functions

$$\left.\frac{\partial^j u}{\partial n^j}\right|_{\Gamma} = \varphi_j \quad (j = 0, 1, \dots, s-1) \tag{*}$$

was equal to the number r, where $2r$ is the order of the differential equation (for

$r > 1$, see ([3,4])). In this case, as we say, weak degeneration occurs, qualitatively not differing from the case $\beta = 0$.

The inequalities discussed below make it possible to justify the correctness of the boundary-value problem (*) with s boundary functions prescribed along the entire boundary Γ, where $r/2 \leqslant s \leqslant r$. A rough explanation of this fact is as follows: if a segment l without endpoints belongs to g, and its endpoints lie on Γ, while at them it does not touch Γ, then, under certain degeneration conditions, the boundary conditions at the endpoints of l in a sense complement one another: at each of the endpoints of l no fewer than $r/2$ boundary values are prescribed, and at the two endpoints together no fewer than r.

The case $r/2 \leqslant s < r$ we call strong degeneration, since in this case the number of boundary functions prescribed on Γ is less than usual. The proofs are carried out by the variational method, by minimizing the integral (9). In the case of homogeneous degeneration, necessary and sufficient conditions for solvability of the problem are indicated in terms of the boundary data. The paper also considers cases of nonhomogeneous degeneration.

2. Let r and s be natural numbers and let α be a real number for which

$$\frac{r}{2} \leq s \leq r, \tag{1}$$

$$s - r - \frac{1}{p} < \alpha < s - r + \frac{1}{p}, \tag{2}$$

$$\|\varphi\|_{L_p(g)} = \left(\int_g |\varphi|^p \, dx\right)^{1/p} \qquad (1 \leq p \leq \infty).$$

Lemma 1. For every function f with finite norm

$$\left\| \frac{d^r f/dx^r}{(x-a)^\alpha (b-x)^\alpha} \right\|_{L_p(a,b)} < \infty \tag{3}$$

the inequality

$$\|f\|_{L_p(a,b)} \leq c \left\{ \sum_{j=0}^{s-1} (|f^{(j)}(a)| + |f^{(j)}(b)|) + \left\| \frac{d^r f/dx^r}{(x-a)^\alpha (b-x)^\alpha} \right\|_{L_p(a,b)} \right\}, \tag{4}$$

holds, where c does not depend on f; here the derivatives $f^{(j)}(x)$ $(j = 0, 1, \dots, s-1)$ are continuous on the (closed) interval $[a, b]$.

We note that, for $s < r$, inequality (4) ceases to be true if in it one omits the terms $f^{(j)}(b)$ and the factor $(b-x)^\alpha$ under the norm sign.

Lemma 2. Let Ω be a domain of n-dimensional space R_n of points $y = (y_1, \dots, y_n) = (\bar{y}, y_n)$, $\bar{y} = (y_1, \dots, y_{n-1})$, defined by the inequalities

$$\psi_1(\bar{y}) < y_n < \psi_2(\bar{y}),$$

$$|\bar{y}|^2 = \sum_{j=1}^{n-1} y_j^2 \le \delta^2, \tag{5}$$

where ψ_1, ψ_2 are continuous functions on the ball (5), which we shall denote by σ.

Then

$$\|f\|_{L_p(\Omega)} \le c \left\{ \sum_{j=0}^{s-1} \left(\left\| f_{y_n}^{(j)}(\bar{y}, \psi_1(\bar{y})) \right\|_{L_p(\sigma)} + \left\| f_{y_n}^{(j)}(\bar{y}, \psi_2(\bar{y})) \right\|_{L_p(\sigma)} \right) + \right.$$

$$\left. + \left\| \frac{\partial^r f / \partial y_n^r}{(y_n - \psi_1(\bar{y}))^\alpha (\psi_2(\bar{y}) - y_n)^\alpha} \right\|_{L_p(\Omega)} \right\}, \tag{6}$$

where c does not depend on $f(y_1, \dots, y_n)$.

Lemma 3. A bounded domain $g \subset R_n$ with continuously differentiable boundary Γ can be covered by a finite number of simplest domains (bridges), joining Γ with Γ, each of which, in the corresponding (for it) rectangular coordinate system, is a domain (bridge) Ω defined in Lemma 2; moreover the functions $y_n = \psi_1(\bar{y})$, $y_n = \psi_2(\bar{y})$ describe the corresponding pieces of the boundary Γ.

Let $g \subset R_n$ be a bounded domain with r-times continuously differentiable boundary Γ, and let $\rho = \rho(x)$ be the distance from the point x to Γ. By definition, $f \in W_{p,\alpha}^{(r)}(g)$ if

$$\|f\|_{W_{p,\alpha}^{(r)}(g)} = \|f\|_{L_p(g)} + \sum \left\| \frac{f^{(k)}}{\rho^\alpha} \right\|_{L_p(g)} < \infty. \tag{7}$$

Theorem 1. For functions $f \in W_{p,\alpha}^{(r)}(g)$ the inequality

$$\|f\|_{L_p(g)} \le c \left(\sum_{0}^{s-1} \left\| \frac{\partial^j f}{\partial n^j} \Big|_\Gamma \right\|_{L_p(\Gamma)} + \sum_{|k|=r} \left\| \frac{f^{(k)}}{\rho^\alpha} \right\|_{L_p(g)} \right), \tag{8}$$

holds, where c does not depend on f.

* The derivative $f^{r-1}(x)$ is assumed to be absolutely continuous inside (a, b).

Proof. On the basis of Lemma 3 it suffices to prove (8) in the case where g is replaced by Ω, and then the question reduces to inequality (6), because: 1) in view of the fact that the straight lines passing through Ω, parallel to the y_n-axis, do not touch Γ, the function $\rho(x)$, as x approaches Γ, has the same order as either $(y_n - \psi_1(\bar{y}))$ or $(\psi_2(\bar{y}) - y_n)$, and 2) the coordinate transformation $(x_1, \dots, x_n) \rightleftarrows (y_1, \dots, y_n)$ is orthogonal, and hence linear and with constant coefficients, and therefore $\partial^r f / \partial y_n^r$ is a linear combination, with constant coefficients, of the derivatives $f^k(x)$ with $|k| = r$.

3. Let $g \subset R_n$ be a bounded domain with boundary Γ of class C^2 and

$$E(f,h) = \int_g \sum_{|k|,|l| \le r} a_{kl}(x) f^{(k)}(x) h^{(l)}(x)\, dx. \tag{9}$$

Here $a_{kl}(x) = a_{lk}(x)$ are functions measurable on g, depending on the nonnegative integer vectors k, l, and satisfying the estimate

$$|a_{kl}(x)| \le \frac{M^2}{\rho^{2\alpha_{kl}}}, \qquad \alpha_{kl} = r + \alpha - \max\{|k|, |l|\}, \tag{10}$$

where $\rho = \rho(x)$ is the distance from x to the boundary Γ of the domain g, and M is a constant. Suppose, moreover, that

$$\sum_{|k|,|l| \le r} a_{kl}(x) \xi_k \xi_l \ge \varkappa \rho^{-2\alpha} \sum_{|k|=r} \xi_k^2, \tag{11}$$

where the ξ_k are numbers depending on the vectors k; $\varkappa$ does not depend on x.

Let a function $\Phi \in W^{(r)}_{p,\alpha}(g)$ be given. Under conditions (1), (2), the boundary functions (see (2))

$$\left.\frac{\partial^j \Phi}{\partial n^j}\right|_\Gamma = \varphi_j \in B_2^{(r+\alpha-j-1/2)}(\Gamma). \tag{12}$$

are meaningful for it.

Let $\mathfrak{M}$ be the class of functions $f \in W^{(r)}_{2,\alpha}(g)$ having the boundary values (12). Consider **problem A** of finding, in the class $\mathfrak{M}$, the minimum of the functional

$$E(f,f) - 2(F,f), \tag{13}$$

where $f \in L_2(g)$ and (F,f) denotes the scalar product in $L_2(g)$.

Theorem 2. Problem A has a unique solution $u \in \mathfrak{M}$. The function u satisfies (in the mean) the boundary conditions (12) and is a generalized solution of the equation

$$L(u) \equiv \sum_{|k|,|l| \le r} (-1)^{(l)} D^{(l)}(a_{kl} u^{(k)}) = F(x) \tag{14}$$

in the sense that

$$E(u,v) - 2(F,v) = 0$$

for every function $v \in W^{(p)}_{2,\alpha}$ possessing zero boundary values (12) ($\equiv v \in \mathfrak{M}_0$).

The proof proceeds by the usual method; the essential role here is played by the Poincaré inequality

$$\|f\|_{L_2(g)} \leq c \sum_{|k|=r} \left\| \frac{f^{(r)}}{\rho^\alpha} \right\|_{L_2(g)}, \tag{15}$$

valid for functions f of the class $\mathfrak{M}_0$. This inequality (15) is a special case of (8) for $p = 2$ and $\partial^j u/\partial n^j\big|_\Gamma = 0$, $j_\Gamma = 0, 1, \dots, s-1$.

4. Let us pass to nonhomogeneous degeneracy. Let r, s_1, and s_2 be natural numbers and α_1, α_2 real numbers for which

$$1 \leqslant s_1 \leqslant r, \qquad 1 \leqslant s_2 \leqslant r, \qquad s_1 + s_2 \geqslant r,$$

$$s_1 - r - \frac{1}{p} < \alpha_1 < s_1 - r + \frac{1}{p},$$

$$s_2 - r - \frac{1}{p} < \alpha_2 < s_2 - r + \frac{1}{p}.$$

Lemma 4. For a function f with finite norm

$$\left\| \frac{f^{(r)}}{(x-a)^{\alpha_1}(b-x)^{\alpha_2}} \right\|_{L_p(a,b)} < \infty$$

the inequality

$$\|f\|_{L_p(a,b)} \leqslant c \left(\sum_{j=0}^{s_1-1} |f^{(j)}(a)| + \sum_{j=0}^{s_2-1} |f^{(j)}(b)| + \left\| \frac{f^{(r)}}{(x-a)^{\alpha_1}(b-x)^{\alpha_2}} \right\|_{L_p(a,b)} \right), \tag{16}$$

holds, where c does not depend on f; moreover, the derivatives $f^{(j)}(x)$ are continuous on (a, b), including the point a (for $j = 0, 1, \dots, s_1 - 1$) and the point b (for $j = 0, 1, \dots, s_2 - 1$).

Starting from Lemma 4, we obtain an analogue of Lemma 2, which makes it possible to treat problems with nonhomogeneous degeneracy in domains admitting a finite covering by bridges of a certain type. Suppose, for example, that $\alpha_1 = \alpha$, $\alpha_2 = 0$, and the boundary Γ of a bounded domain $g \subset R_n$ splits into two parts $\Gamma = \Gamma_1 + \Gamma_2$, and $\rho = \rho(x)$ is the distance from x only to Γ_1, while the domain g is covered by bridges Ω (see Lemma 3), each of which connects either Γ_1 with Γ_2, or Γ_2 with Γ_2. Under conditions (10), (11), (12), where $\rho(x)$ is the distance from x only to Γ_1, one can, arguing as above, justify with the aid of (16) the existence and uniqueness of the solution of equation (15) under the conditions

$$\left.\frac{\partial^j u}{\partial n^j}\right|_{\Gamma_1} = \varphi_j \in B_2^{(r+\alpha-j-1/2)}(\Gamma_1) \qquad (j = 0, 1, \dots, s-1),$$

$$\left.\frac{\partial^j u}{\partial n^j}\right|_{\Gamma_2} = \psi_j \in B_2^{(r-j-1/2)}(\Gamma_2).$$

Lemma 5. Let the integral

$$\int_a^b |f^{(r)}(x)|^p (x-a)^{-\alpha_1 p} (b-x)^{-\alpha_2 p}\, dx$$

be finite, where

$$-\left(1-\frac{1}{p}\right) < \alpha_1 < \frac{1}{p}, \qquad \alpha_2 < -\left(r-\frac{1}{p}\right).$$

Then

$$\int_a^b |f(x)|^p (b-x)^{-\alpha_2 p - rp}\, dx \leqslant c\left(\sum_{j=0}^{r-1} |f^{(j)}(a)|^p + \left\|\frac{f^{(r)}}{(x-a)^{\alpha_1}(b-x)^{\alpha_2}}\right\|_{L_p(a,b)}^p\right),$$

where c does not depend on f; moreover, the derivatives $f^{(j)}(x)$, $j = 0, 1, \dots, r-1$, are continuous on $[a, b)$.

Lemma 5 makes it possible to treat, according to the indicated scheme, problems in which a part of the boundary is entirely free of boundary conditions.

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Note: Figure translations are in progress. See original paper for figures.