Mechanics

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Abstract

Full Text

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On New Particular Solutions of the Problem of the Motion of a Heavy Rigid Body about a Fixed Point

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p. 1. As is known ([1]), the Euler-Poisson equations

$$A\frac{dp}{dt}+(C-B)qr = Mg(y_0\gamma''-z_0\gamma'), \qquad \frac{d\gamma}{dt} = r\gamma'-q\gamma'' \begin{pmatrix} A, B, C;\ p, q, r \\ \gamma, \gamma', \gamma'';\ x_0, y_0, z_0 \end{pmatrix} \tag{1}$$

under the conditions

$$r_0 \text{ large}; \qquad \gamma_0'' \neq 0,\ \pm 1; \quad \lim_{r_0\to\infty}(p_0^2+q_0^2) < \infty \qquad (u_0 = u(t)_{t=0}) \tag{2}$$

can be reduced to a quasilinear autonomous system with two degrees of freedom

$$\frac{d^2p}{d\tau^2}+\omega^2 p = \mu F_1\left(p, \frac{dp}{d\tau}, \gamma, \frac{d\gamma}{d\tau}, \mu\right), \qquad \frac{d^2\gamma}{d\tau^2}+\gamma = \mu F_2\left(p, \frac{dp}{d\tau}, \gamma, \frac{d\gamma}{d\tau}, \mu\right), \tag{3}$$

$$\tau = r_0 t, \qquad \omega^2 = \frac{(A-C)(B-C)}{AB}, \qquad \mu = \frac{1}{r_0}\sqrt{\frac{Mgl}{C}}, \qquad l^2 = x_0^2+y_0^2+z_0^2,$$

possessing the first integral

$$\gamma^2 + \left(\frac{d\gamma}{d\tau}\right)^2 + \mu(\ldots) = 1 - \gamma_0''^2.$$

For system (3), for rational ω ($\omega \neq 1,\ {}^1/2$), periodic solutions were found under the additional condition

$$z_0 \neq 0, \qquad p(t,\mu)_{\mu=0} = q(t,\mu)_{\mu=0} = 0 \tag{4}$$

and the corresponding motions of the rigid body were investigated.

p. 2. It can be shown that the equations of the principal amplitudes for $p(0,0)$ and $q(0,0)$, under the periodicity conditions of system (3), admit, for $\omega = 1/3$, four nonzero solutions

$$p(0,0) = \pm\sqrt{3z_0(C-A)(C-B)}\,R, \qquad q(0,0) = \pm\sqrt{z_0(C-A)A}\,R, \tag{5}$$

$$R^2 = \frac{27Mg\,\gamma_0''\,[3(A+B)-4AB-2C^2]}{2A^3B^2(2C-A-B)}.$$

Choosing as the z-axis of the moving coordinate system an axis for which the inequality $\gamma_0' > 0$ is satisfied, from formulas (5) we obtain that, under the conditions

$$z_0(C-A) > 0, \qquad 8AB - 9C(A+B) + C^2 = 0 \tag{6}$$

there exist periodic solutions of system (1) of period

$$T = \frac{6\pi}{r_0} - \frac{3\pi}{Cr_0^3}\left[Ap^2(0,0) + Bq^2(0,0) + 2Mg\left(z_0\gamma_0'' - x_0\sqrt{1-\gamma_0''^{\,2}}\right)\right] + \frac{1}{r_0^4}(...).$$

The expressions for the Euler angles θ, φ, ψ corresponding to these solutions will be

$$r_0(\theta-\theta_0) = \frac{p(0,0)}{4(C-B)}\left[\theta^{(i)}(t+h) - \theta^{(i)}(h)\right] + \frac{1}{r_0}(...),$$

$$r_0(\psi-\psi_0) = -\frac{Mgz_0}{C}t + \frac{p(0,0)}{4(C-B)\sin\theta_0}\left[\psi^{(i)}(t+h) - \psi^{(i)}(h)\right] + \frac{1}{r_0}(...),$$

$$\varphi - \varphi_0 = r_0 t + \frac{1}{r_0}(...), \tag{7}$$

$$\theta^{(i)}(t) = (A-3B+3C)\cos^{1/3}(4r_0t-\varepsilon_i) - 2(A+3B-3C)\cos^{1/3}(2r_0t+\varepsilon_i),$$

$$\psi^{(i)}(t) = (A - 3B + 3C)\sin^{1/3}(4r_0 t - \varepsilon_i) - 2(A + 3B - 3C)\sin^{1/3}(2r_0 t + \varepsilon_i),$$

$$r_0 h = \varphi_0 - \frac{\pi}{2} + \frac{1}{r_0}(...), \qquad \varepsilon_i = (-1)^i \pi \quad (i = 1, 2).$$

Formulas (7), depending on four arbitrary constants $\theta_0, \varphi_0, \psi_0, r_0$ (r_0 large), make it possible to study the motion of a heavy rigid body with one fixed point in the case under consideration.

3. From the results obtained and the results of paper ([1]) (if one does not consider in it the case $\omega > 1$, corresponding to a rigid body with cavities filled with an ideal incompressible fluid) there follow the following theorems concerning the motion, about a fixed point, of a heavy rigid body set into rapid rotation about the greater or the lesser axis of the inertia ellipsoid.

Theorem 1. *Equations (1) under conditions (2) possess periodic solutions satisfying one of the following relations:*

$$x_0^2 + y_0^2 + z_0^2 \neq 0 \quad \text{for } \omega = 1 \text{ and } \omega \text{ irrational}; \tag{8}$$

$$z_0(A - B) < 0, \quad B - \frac{33}{28}C \neq 0 \quad \text{for } \omega = \frac{1}{3} \tag{9}$$

and the relations (4) for rational ω ($\omega \neq \frac{1}{2}$).

To these periodic solutions there correspond expressions for the Euler angles from formulas (7.4), (7.7) of paper ([1]) and from formula (7), depending on four arbitrary constants $\theta_0, \varphi_0, \psi_0, r_0$ (r_0 large).

For any periodic solution of equations (1) satisfying conditions (2) and the corresponding relations (4), (8), (9), the expressions for the Euler angles are given by formulas (7.4), (7.7) of paper ([1]) and by formulas (7).

Theorem 2. *For the existence in system (1), under relations (2), of periodic solutions ($z_0 \neq 0$, $\omega \neq \frac{1}{2}$) to which: 1) there correspond expressions for the Euler angles depending on five arbitrary constants, or 2) they satisfy the relation*

$$p^2(0,0) + q^2(0,0) \neq 0,$$

it is necessary that the conditions

$$\omega \text{ be rational}, \quad \omega \neq 1, \quad z_0(C - A) > 0$$

be fulfilled.

In the first case, for $\omega = \frac{1}{3}$, it is necessary that the additional condition

$$(A - B)\left(B - \frac{33}{28}C\right) = 0$$

be fulfilled.

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named after M. V. Lomonosov

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CITED LITERATURE

1. Yu. A. Arkhangel' skii, *Prikl. matem. i mekh.*, **27**, no. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.