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Abstract

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MATHEMATICS

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ON THE PROBABILITY THAT A RANDOM POINT FALLS INTO A SMALL NEIGHBORHOOD OF A MOVING MANIFOLD

(Presented by Academician L. S. Pontryagin on 16 V 1964)

Let two objects move in the n-dimensional Euclidean space R^n: a k-dimensional twice differentiable submanifold M, changing its form and position according to the law

$$M = M_s, \tag{1}$$

and a random point of Markov type, whose probability density $p(\sigma, x, s, y)$ satisfies the Kolmogorov equation (^{1})

$$\frac{\partial p}{\partial \sigma} + a^{ij}(\sigma, x)\frac{\partial^2 p}{\partial x^i \partial x^j} + b^i(\sigma, x)\frac{\partial p}{\partial x^i} = 0. \tag{2}$$

Let its n-dimensional ε-neighborhood $U(M)$ move together with M. It is required to compute the probability that the random point enters the neighborhood $U(M)$ during the time interval $\sigma \leq s \leq \tau$.

In the present note the principal term of this probability is found. In the case when the manifold M is simply a controlled point, and $U(M)$ is an n-dimensional ball of radius ε with center at this point, the problem was solved in (^{2},^{3}).

In what follows it is assumed that $n - k \geq 3$.

In order to write down the formula obtained, we first make several constructions.

It is known (cf. (^{2})) that the sought probability $\varphi(\sigma, x, \tau)$ (where x is the initial position of the random point at the time $s = \sigma$) is a solution of equation (2) under the conditions

$$\varphi(\tau, x, \tau) = 0,$$

$$\varphi(\sigma, x, \tau) = 1, \qquad x \in V(M_\sigma), \tag{3}$$

where $V(M)$ is the boundary of the neighborhood $U(M)$.

Through each point m_s of the manifold M_s draw the tangent plane $P(m_s)$. Then choose n linearly independent vectors $e_1, e_2, \dots, e_n$, issuing from the point m_s, so that: a) $e_1, e_2, \dots, e_k$ belong to $P(m_s)$; b) in the coordinate system $\xi^1, \xi^2, \dots, \xi^n$, referred to the basis $e_1, e_2, \dots, e_n$, the differential operator

$$a^{ij}(s, m_s) \frac{\partial^2}{\partial x^i \partial x^j} \tag{4}$$

is written in the form of the Laplace operator

$$\sum_{\nu=1}^{n} \frac{\partial^2}{(\partial \xi^\nu)^2}. \tag{5}$$

The subspace conjugate to $P(m_s)$, spanned by the vectors $e_{k+1}, \dots, e_n$, will be denoted by $Q(m_s)$.

The set of points of the subspace $Q(m_s)$ at a distance ε (in the metric R^n) from the plane $P(m_s)$ is an ellipsoid E_{m_s}. Let its equation in the coordinates ξ be

$$\sum_{i,j=k+1}^{n} c_{ij} \xi^i \xi^j = \varepsilon^2. \tag{6}$$

Obviously, up to small terms of higher order in ε, we have

$$V(M_s) = E_{m_s} \times M_s. \tag{7}$$

In what follows, denote by $w(\xi^{k+1}, \dots, \xi^n)$ the harmonic function that vanishes as $|\xi| \to \infty$ and is equal to unity on the ellipsoid E'_{m_s}, singled out in $Q(m_s)$ by the equation

$$\sum_{i,j=k+1}^{n} c_{ij} \xi^i \xi^j = 1.$$

It is known that w can be represented in the form

$$w = \frac{\alpha(m_s)}{\rho^{n-k-2}} + \Pi(\xi^{k+1}, \dots, \xi^n), \tag{8}$$

where

$$\rho^2 = (\xi^{k+1})^2 + \dots + (\xi^n)^2,$$

$\alpha(m_s)$ is uniquely determined by the dimensions of the ellipsoid E'_{m_s}, and Π is the double-layer potential produced by the ellipsoid E'_{m_s} at the point $(\xi^{k+1}, \dots, \xi^n)$. Differentiating the right- and left-hand sides of relation (8) in the direction ρ and then taking the integral over the surface E'_{m_s}, we readily verify that

$$\int_{E'_{m_s}} \frac{\partial w}{\partial \rho}\, dE'_{m_s} = \frac{4\pi^{(n-k)/2}}{\Gamma[(n-k)/2-1]}\, \alpha(m_s) = \beta(m_s), \tag{9}$$

where Γ is Euler' s gamma function.

We can now formulate the following proposition:

The solution of equation (2) under conditions (3) can be represented in the form

$$\varphi(\sigma, x, \tau) = \varepsilon^{\,n-k-2} \int_\sigma^\tau ds \int_{M_s} p(\sigma, x, s, m_s)\, \beta(m_s)\, dM_s + \omega(\sigma, x, \tau, \varepsilon), \tag{10}$$

where ω has magnitude of order $\varepsilon^{\,n-k-1}$ for any point x separated from the manifold M_σ by a finite distance independent of ε.

In formula (10) the inner integration is carried out over the entire manifold M_s, the volume element in which is induced at each point by the frame $e_1, e_2, \dots, e_k$. It is easy to see that this definition of volume depends only on the coefficients a^{ij} of equation (2) and does not depend on the permissible arbitrariness in the choice of the frame $e_1, \dots, e_k$.

The scheme of proof of the proposition just formulated is as follows. The function

$$\Phi(\sigma, x, \tau) = \varepsilon^{\,n-k-2} \int_\sigma^\tau ds \int_{M_s} p(\sigma, x, s, m_s)\, \beta(m_s)\, dM_s \tag{11}$$

is a solution of equation (2) outside the manifold M_σ and satisfies the first of conditions (3). But it does not satisfy the second, boundary condition (3). It turns out, however, that one can construct an n-dimensional ellipsoidal neighborhood of the manifold M_σ on whose boundary the values of the solutions $\Phi(\sigma, x, \tau)$ and $\varphi(\sigma, x, \tau)$ essentially coincide. Let us construct this neighborhood.

For this, in each subspace $Q(m_\sigma)$ take the ellipsoid $E^*_{m_\sigma}$ singled out by the equation

$$\rho = \varepsilon, \tag{12}$$

and set

$$V^*(M_\sigma) = E^*_{m_\sigma} \times M_\sigma. \tag{13}$$

The surface V^* is precisely the boundary of the neighborhood of the manifold M_σ that we need. We emphasize that, generally speaking, V^* does not coincide with V.

Now, using the known asymptotic representations of the function $p(\sigma, x, s, y)$ (cf., for example, (2)) and carrying out elementary, although rather cumbersome, calculations, we find that for $x_0 \in V^*(M_\sigma)$

$$\Phi(\sigma, x_0, \tau) = \alpha(m_{0\sigma}) + \omega_1(\sigma, x_0, \tau, \varepsilon), \tag{14}$$

where ω_1 is of order $O(1)$ for $\tau - \sigma \leqslant \varepsilon$ and vanishes as $\varepsilon \to 0$ when $\tau - \sigma > \varepsilon$. Here $m_{0\sigma}$ denotes the projection of the point x_0 onto the manifold M_σ in the direction of the plane $Q(m_\sigma)$.

On the other hand, using a method which is a natural analogue of the method of (2), we can write the solution $\varphi(\sigma, x, \tau)$ in a certain special form, from which it is directly seen that

$$\varphi(\sigma, x_0, \tau) = \alpha(m_{0\sigma}) + \omega_2(\sigma, x_0, \tau, \varepsilon), \tag{15}$$

where ω_2 has the same asymptotic character with respect to ε as ω_1.

Comparing relations (14) and (15), it is now not difficult to derive formula (10).

In conclusion we note that in the case $n - k = 2$ the simpler formula is valid

$$\varphi(\sigma, x, \tau) = \frac{2\pi}{|\ln \varepsilon|} \int_\sigma^\tau ds \int_{M_s} p(\sigma, x, s, m_s)\, dM_s + o\left(\frac{1}{|\ln \varepsilon|}\right).$$

It is easily obtained if one uses a result of S. M. Nikol' skii (4).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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