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Abstract

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SOME PROPERTIES OF MODULES IN SPACE

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Here some properties of modules of families of curves and surfaces in space and of conformal capacity are proved.

1. Let G be a simply connected domain of n-dimensional Euclidean space*, on whose boundary ∂G two nonintersecting continua B_0 and B_1, called the bases of the domain, are marked, such that $\partial G-(B_0\cup B_1)$ is connected. The theorems obtained are also valid for a broader class of domains (for example, doubly connected ones with a corresponding choice of bases; see item 6).

By $\gamma(A, B)$ we shall denote a curve $\gamma \in G$ joining two sets A and B; by $\sigma(A, B)$, a hypersurface $\sigma \subset G$ separating in G two sets A and B.

Let, further, $\rho \geqslant 0$ be some function (metric) defined in G. We shall assume that ρ is bounded on every compact set $F \subset G$, that

$$\lim_{P\to P_0} \rho < +\infty,$$

if $P_0 \in \partial G-(B_0\cup B_1)$, and that ρ is integrable over any rectifiable curve $\gamma \subset G$ and over any hypersurface $\sigma \subset G$ measurable in the sense of Hausdorff measure of order $n-1$.

Definition. The function

$$u(P) = \inf_{\gamma(B_0,P)} \int_\gamma \rho\, ds,$$

defined in G, will be called the **potential function**, or **potential**, for ρ.

It is easy to show that for any $\gamma(P_1, P_2)$

$$|u(P_1) - u(P_2)| \leqslant \int_{\gamma(P_1,P_2)} \rho\, ds,$$

so that on any compact set $F \subset G$ $u(P)$ satisfies the Lipschitz condition. Consequently, almost everywhere $u(P)$ is differentiable in every direction,

$$\lim_{P_2\to P_1} \frac{1}{|P_1P_2|} \int_{P_1P_2} \rho\, ds = \rho,$$

where P_1P_2 is the segment joining P_1 and P_2. Thus, from the above inequality we obtain almost everywhere in G

$$|\nabla u| \leqslant \rho. \tag{1}$$

2. Denote by $\{C\}$ the family of all rectifiable curves $\gamma(B_0, B_1)$, and by $\{S\}$ the family of all Hausdorff-measurable hypersurfaces $\sigma(B_0, B_1)$. Put

$$h(\rho) = \inf_{\{\gamma \in C\}} \int_\gamma \rho\, ds, \qquad S(\rho) = \inf_{\sigma \in \{S\}} \int_\sigma \rho^{n-1}\, dS, \qquad V(\rho) = \int_G \rho^n\, dV.$$

For example, if G is a right cylinder, B_0 and B_1 its bases, and $\rho = 1$, then h, S, V will be, respectively, the height, the area of the base, and the volume of this cylinder.

If u is the potential of ρ, then

$$h(\rho) = \inf_{P \in B_1} u(P) \leqslant \sup_{P \in B_1} u(P). \tag{2}$$

It is also easy to show that $\sigma(a) = \{u = a\}$, for any $0 < a < h(\rho)$, separates B_0 and B_1, and therefore

$$S(\rho) \leqslant \int_{\sigma(a)} \rho^{n-1}\, dS. \tag{2'}$$

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* The space is compactified by a point at infinity.

Lemma. If ρ_1 and ρ_2 are two nonnegative metrics, then

$$\int_G \rho_1^{n-1} \rho_2\, dV \geq S(\rho_1) \cdot h(\rho_2); \tag{3}$$

in particular, $V(\rho) \geq S(\rho) \cdot h(\rho)$.

The proof uses the generalized Fubini theorem from (3), by virtue of which

$$\int_G \rho_1^{n-1} |\nabla u|\, dV = \int_0^m d\alpha \int_{u=\alpha} \rho_1^{n-1} dS,$$

where $m = \sup_{P \in B_1} u(P)$, and as u one takes the potential for ρ_2. Hence, from (2) and (2′), we obtain what is required.

Theorem 1. For the equality

$$V(\rho) = S(\rho) \cdot h(\rho) \tag{4}$$

to hold, it is necessary and sufficient that the following conditions be satisfied simultaneously*:

1. $\rho = |\nabla u|$.
2. $u(P) = \text{const}$, if $P \in B_1$.
3. $\displaystyle\int_{\sigma(\alpha)} \rho^{n-1} dS = \text{const}$, if $\sigma(\alpha) = \{u = \alpha\}$.
4. $\displaystyle\int_{\sigma(\alpha)} \rho^{n-1} dS \leq \int_{\sigma} \rho^{n-1} dS$ for any $\sigma \in \{S\}$.

For the proof it is enough to note that, together with (4), the equalities in (1), (2), (2′) are satisfied.

3. The modulus $M\{C\}$ of the family of curves $\{C\}$ and the modulus $M\{S\}$ of the family of hypersurfaces $\{S\}$ can be defined as follows (see (1, 5)):

$$M\{C\} = \inf_{\rho} \frac{V(\rho)}{h^n(\rho)}, \qquad M\{S\} = \inf_{\rho} \frac{V(\rho)}{S^{\frac{n}{n-1}}(\rho)}. \tag{5}$$

Theorem 2. If $V(\rho_0) = S(\rho_0) \cdot h(\rho_0)$, then ρ_0 will be an extremal function both for $\{C\}$ and for $\{S\}$, i.e. we shall have

$$M\{C\} = \frac{V(\rho_0)}{h^n(\rho_0)}, \qquad M\{S\} = \frac{V(\rho_0)}{S^{\frac{n}{n-1}}(\rho_0)}. \tag{6}$$

Indeed, the integral $\displaystyle\int_G \rho\rho_0^{n-1} dV$, with the aid of Hölder' s inequality, can be estimated from above:

$$\int_G \rho\rho_0^{n-1} dV \leq V^{\frac{1}{n}}(\rho) \cdot V^{\frac{n-1}{n}}(\rho_0), \tag{7}$$

and with the aid of the lemma—from below. After simple transformations we obtain

$$\frac{V^{\frac{1}{n}}(\rho_0)}{h(\rho_0)} \leq \frac{V^{\frac{1}{n}}(\rho)}{h(\rho)},$$

whence the first of the equalities (6) follows. The second is proved analogously with the aid of estimates of the integral $\displaystyle\int_G \rho_0\rho^{n-1} dV$.

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* Here and below it is assumed that $\rho > 0$ almost everywhere. Two functions that coincide almost everywhere are not distinguished.

If, together with ρ_0, another function ρ satisfies equation (4), then in this reasoning everywhere, including in (7), there will be equalities, whence it follows that $\rho = \lambda\rho_0$, $\lambda = \text{const}$. Thus, by equation (4), ρ is determined uniquely up to a numerical factor.

Corollary 1. $M\{C\}$ and $M\{S\}$ are related by the relation*

$$M\{C\}M^{n-1}\{S\} = 1. \tag{8}$$

This is obtained at once from (6) and (4). From this corollary and from (5) follows

Corollary 2. The estimates hold

$$\frac{S^n(\rho)}{V^{n-1}(\rho)} \leqslant M\{C\} \leqslant \frac{V(\rho)}{h^n(\rho)}; \tag{9}$$

$$\frac{h^{\frac{n}{n-1}}(\rho)}{V^{\frac{n}{n-1}}(\rho)} \leqslant M\{S\} \leqslant \frac{V(\rho)}{S^{\frac{n}{n-1}}(\rho)}, \tag{9'}$$

in each of which equality is attained only for the extremal function.

4. The conformal capacity $\Gamma(G)$ is introduced by the formula

$$\Gamma(G) = \inf_u \int_G |\nabla u|^n dV,$$

where the greatest lower bound is taken over all continuously differentiable functions such that

$$u(P) = 0 \quad \text{for } P \in B_0; \qquad u(P) = 1 \quad \text{for } P \in B_1. \tag{10}$$

If for one of such functions $\Gamma(G) = \int_G |\nabla u|^n dV$, then u satisfies the Euler equation for the functional $\int_G |\nabla u|^n dV$:

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = 0 \tag{11}$$

and the boundary conditions

$$u(P) = 0, \quad P \in B_0; \qquad u(P) = 1, \quad P \in B_1;$$

$$\frac{\partial u}{\partial \nu} = 0, \quad P \in \partial G - (B_0 \cup B_1) \tag{12}$$

with the corresponding assumptions on the existence of the normal ν on the boundary ∂G.

Theorem 3. *The modulus of the family of curves* $\{C\}$ *is equal to the conformal capacity of the domain* G*:* $M\{C\} = \Gamma\{G\}$.

Indeed, putting $\rho = |\nabla u|$, where u satisfies conditions (10), we obtain $M\{C\} \leqslant \Gamma(G)$. If, however, $\rho \geqslant 0$ is some metric and u is its potential, then, taking $u_0 = u/h(\rho)$, we obtain, in view of (1), $\Gamma(G) \leqslant M\{C\}$. The theorem is proved.

Theorem 4. *If* ρ_0 *is an extremal metric for* $\{C\}$ *(respectively* $\{S\}$*), then it is extremal also for* $\{S\}$ *(respectively* $\{C\}$*), and*

$$V(\rho_0) = S(\rho_0) \cdot h(\rho_0).$$

Indeed, let u be the potential of ρ_0. It then follows from (11) and (12) that

$$V(|\nabla u|) = S(|\nabla u|) \cdot h(|\nabla u|), \tag{13}$$

and the application of theorem 2 completes the proof.

Thus, the extremal metric both for $\{C\}$ and for $\{S\}$ is determined uniquely up to a numerical factor. We shall simply call such a metric extremal (for G).

Remark. Equation (13) holds if u satisfies equation (11) and conditions (12) and if every level surface of u separates B_0 and B_1. Thus (13) may be regarded as an integral form for (11) and (12).

5. The following two theorems extend to space the known principles of Grötzsch (see also ([1])).

Theorem 5. *Let some surface* $\sigma(B_0, B_1)$ *divide* G *into two parts* G_1 *and* G_2*, taken with bases* B_0 *and* σ*,* σ *and* B_1*. Then*

$$\Gamma^{\frac{1}{1-n}}(G_1) + \Gamma^{\frac{1}{1-n}}(G_2) \leq \Gamma^{\frac{1}{1-n}}(G),$$

and equality holds only in the case when on the indicated surface $u = \text{const}$*, where* u *is the potential of the extremal function of the domain* G*.*

For the proof it is necessary to estimate $\Gamma(G_1)$ and $\Gamma(G_2)$ from below by means of (9), taking ρ extremal for G, and to use property 3 of Theorem 1.

Theorem 6. *Let some piecewise-smooth surface* σ *divide* G *into two parts* $\overline{G}_1$ *and* $\overline{G}_2$ *and intersect both bases of the domain* G*, also dividing each of them into two parts, considered as bases for* $\overline{G}_1$ *and* $\overline{G}_2$*. Then*

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[*] This relation and theorem 3 were obtained by B. V. Shabat by another method (unpublished).

$$\Gamma(\overline{G}_1) + \Gamma(\overline{G}_2) \le \Gamma(G),$$

and equality holds only in the case when on the indicated surface $\partial u/\partial \nu \equiv 0$, *where* ν *is the normal and* u *is the potential of the extremal function.*

For the proof it is necessary to estimate $\Gamma(\overline{G}_1)$ and $\Gamma(\overline{G}_2)$ from above by means of (9), taking ρ extremal for G, and to use property 2 of Theorem 1.

6. Let $P^* = f(P)$ be a conformal mapping of G onto G^*.

Theorem 7. *If* $u(P)$ *satisfies equation (11), then* $u^*(P^*) = u(P)$ *also satisfies this equation.*

To verify this, one must, taking G sufficiently small so as to have (12), use the equalities

$$h(|\nabla u^*|) = h(|\nabla u|), \quad S(|\nabla u^*|) = S(|\nabla u|), \quad V(|\nabla u^*|) = V(|\nabla u|),$$

equation (13), and the remark to Theorem 4.

We shall denote by an asterisk the image of a set under the transformation of symmetry with respect to an $(n-1)$-dimensional sphere.

Theorem 8. *Let* B_1 *be a piece of an* $(n-1)$*-dimensional sphere. Then, if* u *is the potential of the function extremal for* G*, the function*

$$\rho(P) = \begin{cases} |\nabla u|, & \text{if } P \in G, \\ |\nabla u^*|, & \text{if } P \in G^*, \quad u^*(P^*) = u(P), \end{cases}$$

will be extremal for the domain $G + G^*$*, taken with bases* B_0 *and* B_0^**. Moreover,*

$$\Gamma^{\frac{1}{1-n}}(G + G^*) = 2\Gamma^{\frac{1}{1-n}}(G). \tag{14}$$

Indeed, the extremality of ρ for $G + G^*$ follows from Theorem 7 and the remark to Theorem 4, while formula (14) follows from Theorem 5.

If, for example, G is a Grötzsch ring, then $G + G^*$ will be a Teichmüller ring, and (14) will give a relation between the moduli of these rings, which was obtained by another method in (^{2}) by Gehring. As the bases of G one must choose the components of the boundary.

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References

[1] B. V. Shabat, DAN, 130, No. 6 (1960).
[2] F. W. Gehring, Trans. Am. Math. Soc., 101, 499 (1961).
[3] G. Federer, Trans. Am. Math. Soc., 93, 426 (1959).
[4] F. W. Gehring, Trans. Am. Math. Soc., 103, 353 (1962).

Note: Figure translations are in progress. See original paper for figures.

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