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CYBERNETICS AND CONTROL THEORY

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

E. I. NECHIPORUK

ON THE SYNTHESIS OF LOGICAL NETWORKS IN INCOMPLETE AND DEGENERATE BASES

(Presented by Academician V. I. Smirnov on 22 XI 1963)

1°. A fundamental problem of cybernetics is the construction of control systems with prescribed functional properties under optimal use of the initial means (basis). A special case of this problem is the problem of realizing Boolean functions by logical networks.

Let us consider the realization of Boolean functions by logical networks of two types: circuits composed of functional elements and superpositions (¹⁻³).

To each basis element we assign a nonnegative real number, called its **weight**. Basis elements to which weight zero is assigned, and the corresponding basic Boolean functions, will be called **zero**; basis elements to which a positive weight is assigned, and the corresponding basic Boolean functions, will be called **positive**. The set of zero functions of a basis \mathfrak{F} will be called its **zero part** and denoted by $\mathfrak{Z}(\mathfrak{F})$; the set of positive functions of the basis will be called its **positive part**. If \mathfrak{M} is a set of Boolean functions, then by $\mathfrak{F}_{\mathfrak{M}}$ we denote the closed class of Boolean functions generated by the set \mathfrak{M} . We may assume that every function from $\mathfrak{F}_{\mathfrak{Z}(\mathfrak{F})}$ is represented in the zero part of the basis up to a renaming of variables, and that the zero and positive parts of the basis do not intersect. The pair $(\mathfrak{F}_{\mathfrak{Z}(\mathfrak{F})}, \mathfrak{F}_{\mathfrak{P}(\mathfrak{F})})$ will be called the **type** of the basis \mathfrak{F} . We number the positive basis elements by the numbers $k = 1, 2, \dots$ and denote them by E_k ; the weight of the element E_k is denoted by Ω_k ; the number of inputs of the element E_k is denoted by m_k ; the basic Boolean function corresponding to the element E_k is denoted by $e_k(Y_1, \dots, Y_{m_k})$. We assume that every basic function depends essentially on all its variables. A basis is called **finite** if its positive part is finite, and **infinite** otherwise. A basis is called **complete** if every Boolean function can be realized in it, and **incomplete** otherwise. A basis is called **nondegenerate** if it contains no zero functions depending on more than one variable, and **degenerate** otherwise.

The function realized by a network \mathfrak{A} is denoted by $\chi_{\mathfrak{A}}$. The weight of the network \mathfrak{A} is defined as the sum of the weights of all basis elements entering into it and is denoted by $\Omega_{\mathfrak{F}}^{(1)}(\mathfrak{A})$. The functions

$$\Omega_{\mathfrak{F}}^{(2)}(f) = \min_{\substack{\mathfrak{A} \\ (\chi_{\mathfrak{A}}=f)}} \Omega_{\mathfrak{F}}^{(1)}(\mathfrak{A}), \quad \Omega_{\mathfrak{F}}(n) = \max_{\substack{f(x_1, \dots, x_n) \\ (f(x_1, \dots, x_n) \in \mathfrak{F}_{\mathfrak{F}})}} \Omega_{\mathfrak{F}}^{(2)}(f)$$

are called **Shannon functions**.

The problem consists in finding asymptotic estimates for the function $\Omega_{\mathfrak{F}}(n)$. In the works of O. B. Lupanov the indicated problem was completely solved for arbitrary finite, complete, and nondegenerate bases ⁽¹⁻³⁾. Below we give a solution of the problem for certain bases that do not satisfy the conditions of completeness and nondegeneracy. A number of particular results in this direction

was obtained by the author earlier ⁽⁵⁾. It is established that changing the basis while preserving its zero part has no essential effect on the methods of synthesis; thus, for some incomplete nondegenerate bases the solution reduces to the constructions of O. B. Lupanov. When the zero part of the basis is changed, however, the solution of the problem requires essentially different approaches in different cases.

2°. Let \mathfrak{F} be an arbitrary finite nondegenerate basis and $\mathfrak{K}_{\mathfrak{F}} = C_j$, where $1 \leq j \leq 4^*$. Put

$$\mathfrak{C}_{\mathfrak{F}}^I = \min_{(m_k > 1)} \frac{\Omega_k}{m_k - 1}.$$

Theorem 1. *For circuits there is the asymptotic estimate*

$$\Omega_{\mathfrak{F}}(n) \sim \mathfrak{C}_{\mathfrak{F}}^I \frac{2^n}{n}.$$

3°. Let \mathfrak{F} be an arbitrary finite basis of type (L_i, C_j) , where $1 \leq i \leq 5$, $1 \leq j \leq 4^{**}$.

Denote by $m(\varphi, \mathfrak{D}, \Lambda)$ the number of essential variables of the function

$$\varphi(\mathfrak{D}\tilde{y}) \oplus \Lambda(\tilde{y}),$$

where \mathfrak{D} is a nonsingular square matrix, and Λ is a linear function without a constant term. The number

$$\min_{\mathfrak{D}, \Lambda} m(\varphi, \mathfrak{D}, \Lambda)$$

is called the **linear dimension** of the function $\varphi(\tilde{y})$. We denote the linear dimension of the function e_k by d_k .

Let the positive part of the basis \mathfrak{F} contain K basis functions. Put

$$\mathfrak{C}_{\mathfrak{F}}^{\text{II}} = \min_{\substack{k \\ (d_k > 0)}} \frac{\Omega_k}{d_k},$$

$$\mathfrak{C}_{\mathfrak{F}}^{\text{III}} = \left(\max_{\{\beta_k\}_{1^K}} \sum_{1 \leq k < K} d_k \left(\frac{\beta_k^2}{2\Omega_k^2} + \sum_{1 \leq l < k} \frac{\beta_k \beta_l}{\Omega_k \Omega_l} \right) \right)^{-1/2},$$

where the numbering of the basis functions is such that $d_1 \leq d_2 \leq \dots \leq d_k$, and the maximum is taken over the simplex determined by the conditions

$$0 \leq \beta_1, \dots, \beta_K \leq 1, \quad \sum_{1 \leq k < K} \beta_k \leq 1.$$

Theorem 2. For superpositions there is the asymptotic estimate

$$\Omega_{\mathfrak{F}}(n) \sim \mathfrak{C}_{\mathfrak{F}}^{\text{II}} \frac{2^n}{n}.$$

Theorem 3. For circuits there is the asymptotic estimate

$$\Omega_{\mathfrak{F}}(n) \sim \mathfrak{C}_{\mathfrak{F}}^{\text{III}} 2^{n/2}.$$

* According to Post ⁽⁴⁾, the class C_1 consists of all functions; the class C_2 is defined by the equality $\varphi(\tilde{1}) = 1$; the class C_3 is defined by the equality $\varphi(\tilde{0}) = 0$; the class C_4 is defined by the equalities $\varphi(\tilde{1}) = 1$, $\varphi(\tilde{0}) = 0$ (the symbols $\tilde{0}$ and $\tilde{1}$ denote vectors all of whose components are zeros and ones, respectively).

** According to Post, L_1, \dots, L_5 are the classes of linear functions. The condition $L_i \subset C_j$ is understood.

4°. Let \mathfrak{F} be an arbitrary, possibly infinite, basis of type (A_i, C_j) , where $1 \leq i \leq 4$, $1 \leq j \leq 4$ *, such that

$$\inf_{\substack{k \\ (m_k > 0)}} \frac{\Omega_k}{m_k} > 0.$$

A set $\mathfrak{S} = \{\tilde{\sigma}^i\}_0^n$ of Boolean vectors of length n is called a chain if $n > 0$ and each vector is obtained from the preceding one by replacing zero by one in some component. Let φ be an arbitrary Boolean function of n arguments. The vector $\tilde{\sigma}^i$ ($1 \leq i < n$) is called an inversion node of the pair (φ, \mathfrak{S}) if $\varphi(\tilde{\sigma}^i) = 1$, $\varphi(\tilde{\sigma}^{i+1}) = 0$. Denote by $M_{\mathfrak{S}}(\varphi)$ the number of inversion nodes of the pair (φ, \mathfrak{S}) ; put

$$M(\varphi) = \max_{\mathfrak{E}} M_{\mathfrak{E}}(\varphi).$$

Put

$$\mathfrak{C}_{\mathfrak{F}}^{\text{IV}} = \inf_{\substack{\varphi \\ (M(\varphi) > 0)}} \frac{\Omega_{\mathfrak{F}}^{(2)}(\varphi)}{M(\varphi)},$$

where the infimum is taken over all Boolean functions.

Theorem 4. As $M(f_m) \rightarrow \infty$, for superpositions the following asymptotic estimate holds

$$\Omega_{\mathfrak{F}}^{(2)}(f_m) \sim \mathfrak{C}_{\mathfrak{F}}^{\text{IV}} M(f_m).$$

Corollary.

$$\Omega_{\mathfrak{F}}(n) \sim 1/2 \mathfrak{C}_{\mathfrak{F}}^{\text{IV}} n.$$

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* According to Post, A_1, \dots, A_4 are classes of monotone functions differing only by the inclusion of constants. The condition $A_i \subset C_j$ is assumed.

Note: Figure translations are in progress. See original paper for figures.

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