ON INTEGRATION WITH ACCURACY CONTROL

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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ON INTEGRATION WITH ACCURACY CONTROL

(Presented by Academician M. V. Keldysh, 27 VII 1963)

1°. For integrating $f(x)$ on $[-1,+1]$, quadrature formulas of the form

$$A(f) = C_1 f(x_1) + \dots + C_n f(x_n) \qquad (x_i \neq x_k \text{ for } i \neq k) \tag{1}$$

are considered. By the **accuracy** of a quadrature A we shall mean the greatest natural number $T(A)$ for which all polynomials of degree $T(A)$ are integrated exactly by formula (1). The number $(A) = n$ will be called the **cost** of A. Among quadratures of cost n, the Gaussian quadrature Γ_n has the maximal accuracy $2n-1$. The accuracy of other quadratures is $\leqslant 2n-2$.

2°. Let a pair of quadratures A, B be given. This means that A and B are different quadratures and $T(A) \leqslant T(B)$. By the **cost** (A, B) of the pair A, B we shall mean the total number of distinct points x_i occurring in A and B, and by the **accuracy** of the pair the number $T(A,B) = T(A)$.

3°. **Lemma.** Let $x_1, \dots, x_k$ be arbitrary distinct points. There exists a quadrature D

$$D(f) = d_1 f(x_1) + \dots + d_k f(x_k) \tag{2}$$

of accuracy $\geqslant k-1$. Among quadratures of accuracy $\geqslant k-1$, it is unique.

Indeed, for $f(x) = x^s$ $(s = 0, 1, \dots, k-1)$, the determinant of the system

$$D(x^s) = 2 : (s+1) \qquad (s = 0, 1, \dots, k-1) \tag{2a}$$

is a Vandermonde determinant, different from zero.

4°. **Theorem 1.** Let A, B be a pair of quadratures. Then

$$(A, B) \geqslant T(A, B) + 2.$$

Indeed, let $x_s\,[s = 1, 2, \dots, N = (A, B)]$ be all the points entering into the formulas A and B. The quadratures A and B have the form (2). By the lemma, they are identical if $T(A) \geqslant (A, B) - 1$.

5°. For integration with accuracy control, with permitted cost $N = 2n + 1$*, we shall call **optimal** a pair A, B with maximal accuracy $T(A, B)$, minimal cost $(A, B) \leqslant N$ among all such pairs, with minimal (A) among these, and with maximal $T(B)$ among these. According to 1° and 4° we have $A \equiv \Gamma_n$. Therefore $T(A, B) = 2n - 1 = N - 2$. We construct a quadrature K_n which, together with Γ_n, gives an optimal pair. Let L be the Legendre polynomial of degree n with leading coefficient 1. Let I_p be the $(n + p)$-th moment of L. Let $t = 0$ for even n, and $t = 1$ for odd n. Put

$$b_0 = 1; \qquad b_{2k} = -(b_0 I_{2k} + b_2 I_{2k-2} + \dots + b_{2k-2} I_2) : I_0; \tag{3}$$

$$\varphi(x) = b_0 x^{n+1} + b_2 x^{n-1} + \dots + b_{n+t} x^{1-t}. \tag{4}$$

Then $F(x) = L(x) \cdot \varphi(x)$ is orthogonal to x^s $(s = 0, 1, \dots, n+1-t)$. The function $F(x)$ has $2n + 1$ distinct roots $\{\beta_s\}$ on $[-1, +1]$**; moreover–

* For an even permitted cost, I do not know how to approach this problem reasonably.

** For arbitrary n I do not know how to prove this, but up to $n \leqslant 40$ the roots have been computed directly.

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
2	0.0370	0.0000	0.0989	$r_4 = 0.444 \cdot$
	8995	0000	8989	$10^0 r_8 =$
	01142113	00005000	89902454	$0.163 \cdot$
	2486	0000	5454	$10^{-1} r_{10} =$
	54055000	00000000	54553111	$0.298 \cdot 10^{-1}$
	0000 0000	0000 0000	1111 1110	
3	0.0197	0.0000	0.0523	$r_6 = 0.160 \cdot$
	5436	0000	2811	$10^0 r_{12} =$
	56461127	00002777	30131342	$0.182 \cdot$
	0166	7777	4404	$10^{-2} r_{14} =$
	53792828	77780000	49342006	$0.161 \cdot$
	7812	0000	9807	$10^{-1} r_{16} =$
	53275000	00004444	93882254	$0.102 \cdot 10^{-1}$
	0000 0000	4444 4444	8586 9330	

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
4	0.0117 1287 46319514 9308 45411798 5689 12523300 0994 82085000 0000 0000	0.0000 0000 00001739 7391 22590000 0000 00003260 7257 74310000 0000 0000	0.0314 6813 68330858 2680 26661333 9917 02261634 2459 48011732 2149 0944	$r_8 = 0.523 \cdot 10^{-1}$ $r_{14} = 0.755 \cdot 10^{-4}$ $r_{16} = 0.491 \cdot 10^{-3}$ $r_{18} = 0.152 \cdot 10^{-2}$ $r_{20} = 0.330 \cdot 10^{-2}$
5	0.0079 5731 99530786 5805 70311229 1663 67152307 0534 49473601 8979 34195000 0000 0000	0.0000 0000 00001184 0634 42280000 0000 00002393 0438 52500000 0000 00002841 4444 4444	0.0212 9101 83760565 4165 53110934 0039 82781205 2016 96141364 2482 89561441 9370 8930	$r_{10} = 0.162 \cdot 10^{-1}$ $r_{18} = 0.466 \cdot 10^{-5}$ $r_{20} = 0.582 \cdot 10^{-4}$ $r_{22} = 0.207 \cdot 10^{-3}$ $r_{24} = 0.720 \cdot 10^{-3}$ $r_{26} = 0.106 \cdot 10^{-2}$
6	0.0055 4839 86940397 6524 28980893 1332 95671693 3530 67672634 4849 73623806 9040 69585000 0000 0000	0.0000 0000 00000856 6224 61900000 0000 00001803 8978 65240000 0000 000002339 5696 72860000 0000 0000	0.0151 9807 266004418 4722 02230686 6030 23170905 9599 71621106 0690 46381168 8543 20581205 3629 0088	$r_{12} = 0.480 \cdot 10^{-2}$ $r_{20} = 0.634 \cdot 10^{-6}$ $r_{22} = 0.446 \cdot 10^{-5}$ $r_{24} = 0.800 \cdot 10^{-4}$ $r_{26} = 0.535 \cdot 10^{-4}$ $r_{28} = 0.129 \cdot 10^{-3}$ $r_{30} = 0.266 \cdot 10^{-3}$

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
7	0.0044 7231 44400254 6604 38290675 3787 83201239 2493 22002069 5638 22662970 7742 43113961 0725 24965000 0000 0000	0.0000 0000 00000647 4248 30840000 0000 00001398 5263 57450000 0000 00001502 5253 00002089 7959 1836	0.0114 6766 10050315 4604 63150523 9500 51610739 2066 95980845 0236 33200951 7528 90321022 1617 00381047 4107 0542	$r_{14} = 0.139 \cdot 10^{-2}$ $r_{16} = 0.588 \cdot 10^{-2}$ $r_{24} = 0.718 \cdot 10^{-6}$ $r_{26} = 0.515 \cdot 10^{-5}$ $r_{28} = 0.238 \cdot 10^{-5}$ $r_{30} = 0.756 \cdot 10^{-5}$ $r_{32} = 0.995 \cdot 10^{-4}$ $r_{34} = 0.434 \cdot 10^{-4}$
8	0.0033 1006 20590198 5507 17510529 3994 65761016 0565 24531658 6226 45292372 3379 50423196 4945 10364082 8652 77525000 0000 0000	0.0000 0000 00000506 1426 81450000 0000 00001111 1051 27270000 0000 00001568 5332 29390000 0000 00001813 4189 16890000 0000 0000	0.0089 1119 16600247 1969 75010412 4114 94660585 2318 54530681 8513 46220783 2630 30840860 3530 42780910 2004 23340922 2320 2872	$r_{16} = 0.396 \cdot 10^{-3}$ $r_{26} = 0.645 \cdot 10^{-8}$ $r_{28} = 0.539 \cdot 10^{-7}$ $r_{30} = 0.242 \cdot 10^{-6}$ $r_{32} = 0.842 \cdot 10^{-6}$ $r_{34} = 0.229 \cdot 10^{-5}$ $r_{36} = 0.540 \cdot 10^{-5}$ $r_{38} = 0.110 \cdot 10^{-4}$ $r_{40} = 0.223 \cdot 10^{-4}$

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
9	0.0026	0.0000	0.0071	$r_{18} = 0.111 \cdot$
	6091	0000	5238	$10^{-3} r_{30} =$
	96610159	00000406	78220198	$0.738 \cdot$
	1988	3779	1594	$10^{-9} r_{32} =$
	02460425	41810000	75800332	$0.667 \cdot$
	1824	0000	9907	$10^{-8} r_{34} =$
	63750819	00000909	79700475	$0.310 \cdot$
	4841	3040	3955	$10^{-7} r_{36} =$
	43331327	33370000	00450558	$0.118 \cdot$
	5661	0000	9456	$10^{-6} r_{38} =$
	74081933	00001303	73420650	$0.343 \cdot$
	1428	0534	0070	$10^{-6} r_{40} =$
	36502622	82010000	34280726	$0.867 \cdot$
	6883	0000	1979	$10^{-6} r_{42} =$
	44243378	00001567	41920776	$0.191 \cdot$
	7282	7535	8468	$10^{-5} r_{44} =$
	82984178	88380000	39840814	$0.388 \cdot$
	8821	0000	4311	$10^{-5} r_{46} =$
	81935000	00001651	37200824	$0.733 \cdot 10^{-5}$
	0000 0000	1967 7502	4800 6414	

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
10	0.0021	0.0000	0.0058	$r_{20} = 0.307 \cdot$
	7741	0000	4731	$10^{-4} r_{32} =$
	84870130	00000333	94340162	$0.726 \cdot$
	4675	3567	7908	$10^{-10} r_{34} =$
	52410345	21540000	11540272	$0.107 \cdot$
	2473	0000	7987	$10^{-8} r_{36} =$
	21380674	00000747	82270375	$0.374 \cdot$
	6831	2567	1983	$10^{-8} r_{38} =$
	66561095	45750000	74050465	$0.142 \cdot$
	9113	0000	6272	$10^{-7} r_{40} =$
	67071602	00001095	72920541	$0.435 \cdot$
	9251	4318	9357	$10^{-7} r_{42} =$
	58502188	12580000	94010612	$0.114 \cdot$
	1541	0000	4795	$10^{-6} r_{44} =$
	36562833	00001346	81310673	$0.262 \cdot$
	0230	5835	5460	$10^{-6} r_{46} =$
	29953528	96550000	86560713	$0.569 \cdot$
	0358	0000	8796	$10^{-6} r_{48} =$
	65494255	00001477	92890738	$0.112 \cdot$
	6283	6211	8358	$10^{-5} r_{50} =$
	05095000	23580000	74510741	$0.202 \cdot 10^{-5}$
	0000 0000	0000 0000	2227 7000	

No.	Nodes	Gauss quadrature weight	Exact quadrature weight	Relative error in the last significant digit
11	0.0018	0.0000	0.0048	$r_{22} = 0.842 \cdot$
	1519	0000	8272	$10^{-5} r_{24} =$
	30550108	00000278	05230135	$0.525 \cdot$
	8567	3428	7827	$10^{-4} r_{26} =$
	09270291	35580000	73410229	$0.184 \cdot$
	6444	0000	1468	$10^{-3} r_{36} =$
	57710564	00000623	92820325	$0.177 \cdot$
	8680	9018	4871	$10^{-11} r_{38} =$
	41110979	74730000	32750393	$0.881 \cdot$
	7172	0000	2323	$10^{-10} r_{40} =$
	16721349	00000931	96560464	$0.496 \cdot$
	2399	4510	7654	$10^{-9} r_{42} =$
	72131847	94640000	92980529	$0.200 \cdot$
	0023	0000	3603	$10^{-8} r_{44} =$
	99192404	00001165	72410583	$0.610 \cdot$
	4353	3888	2975	$10^{-8} r_{46} =$
	87973010	22260000	71310629	$0.179 \cdot$
	2792	0000	7939	$10^{-7} r_{48} =$
	95243652	00001314	95500656	$0.437 \cdot$
	2842	0227	4034	$10^{-7} r_{50} =$
	20244319	22550000	21150672	$0.969 \cdot$
	4390	0000	9678	$10^{-7} r_{52} =$
	96005000	00001364	64000628	$0.199 \cdot 10^{-6}$
	0000 0000	6254 3390	8889 7350	

whose roots are symmetric with respect to the point $\beta_{n+1} = 0$. Put

$$K_n(f) = d_1 f(\beta_1) + \ldots + d_{2n+1} f(\beta_{2n+1}). \tag{5}$$

By the lemma we choose d_i so that $T(K_n') \geq 2n$. Such a quadrature formula is unique and $d_i = d_{2n+2-i}$. Hence it follows that $T(K_n) \geq 2n + 1$.

6°. Theorem 2. $T(K_n) = 3n + 1$ for even n, and $T(K_n) = 3n + 2$ for odd n.

Indeed, $F(x)$ is orthogonal to x^s $(s = 0, 1, \ldots, n + 1 - t)$. Further, $\{\beta_i\}$ are the roots of $F(x)$. Hence K_n is exact for certain polynomials of degree $2n+1+s$ $(s = 0, 1, \ldots, n + 1 - t)$ with leading coefficient 1, namely for $F(x) \cdot x^s$. Consequently, K_n is exact for all polynomials up to degree $(3n + 1 + t)$.

n	Nodes	Weights of Gaussian quadrature	Weights of the refining quadrature	Relative error of the refining quadrature
12	0.0015 3303 873[illegible]	[illegible]	[illegible]	[illegible] 0.230 · $10^{-5} K_{26} =$ 0.155 · $10^{-4} K_{28} =$ 0.520 · $10^{-4} K_{38} =$ 0.866 · $10^{-12} K_{40} =$ 0.972 · $10^{-11} K_{42} =$ 0.586 · $10^{-10} K_{44} =$ 0.205 · $10^{-9} K_{46} =$ 0.855 · $10^{-9} K_{48} =$ 0.242 · $10^{-8} K_{50} =$ 0.632 · $10^{-8} K_{52} =$ 0.145 · $10^{-7} K_{54} =$ 0.311 · $10^{-7} K_{56} =$ 0.621 · 10^{-7}

n	Nodes	Weights of Gaussian quadrature	Weights of the refining quadrature	Relative error of the refining quadrature
13	0.0013 1691 150[illegible]	[illegible]	[illegible]	[illegible] K_{26} = 0.621 · 10^{-6} K_{28} = 0.450 · 10^{-5} K_{30} = 0.180 · 10^{-4} K_{42} = 0.101 · 10^{-12} K_{44} = 0.121 · 10^{-11} K_{46} = 0.771 · 10^{-11} K_{48} = 0.371 · 10^{-10} K_{50} = 0.125 · 10^{-9} K_{52} = 0.379 · 10^{-9} K_{54} = 0.101 · 10^{-8} K_{56} = 0.245 · 10^{-8} K_{58} = 0.541 · 10^{-8} K_{60} = 0.112 · 10^{-7} K_{62} = 0.217 · 10^{-7}

n	Nodes	Weights of Gaussian quadrature	Weights of the refining quadrature	Relative error of the refining quadrature
14	0.0011 3970 312[illegible]	[illegible]	[illegible]	[illegible] $K_{28} =$ $0.167 \cdot 10^{-6}$ $K_{30} = 0.129 \cdot 10^{-5}$ $K_{32} = 0.548 \cdot 10^{-5}$ $K_{44} = 0.107 \cdot 10^{-13}$ $K_{46} = 0.132 \cdot 10^{-12}$ $K_{48} = 0.927 \cdot 10^{-12}$ $K_{50} = 0.440 \cdot 10^{-11}$ $K_{52} = 0.166 \cdot 10^{-10}$ $K_{54} = 0.590 \cdot 10^{-10}$ $K_{56} = 0.148 \cdot 10^{-9}$ $K_{58} = 0.371 \cdot 10^{-9}$ $K_{60} = 0.854 \cdot 10^{-9}$ $K_{62} = 0.184 \cdot 10^{-8}$ $K_{64} = 0.368 \cdot 10^{-8}$ $K_{66} = 0.704 \cdot 10^{-8}$

n	Nodes	Weights of Gaussian quadrature	Weights of the refining quadrature	Relative error of the refining quadrature
15	0.0009 9885 665[illegible]	[illegible]	[illegible]	[illegible] K_{30} = 0.446 · 10^{-7} K_{32} = 0.366 · 10^{-6} K_{34} = 0.166 · 10^{-5} K_{48} = 0.555 · 10^{-14} K_{50} = 0.74 · 10^{-13} K_{52} = 0.422 · 10^{-12} K_{54} = 0.165 · 10^{-11} K_{56} = 0.240 · 10^{-11} K_{58} = 0.797 · 10^{-11} K_{60} = 0.232 · 10^{-10} K_{62} = 0.385 · 10^{-10} K_{64} = 0.145 · 10^{-9} K_{66} = 0.322 · 10^{-9} K_{68} = 0.669 · 10^{-9} K_{70} = 0.132 · 10^{-8} K_{72} = 0.247 · 10^{-8}

n	Nodes	Weights of Gaussian quadrature	Weights of the refining quadrature	Relative error of the refining quadrature
16	0.0008 8036 292[illegible]	[illegible]	[illegible]	[illegible] K_{32} = 0.119 · 10^{-7} K_{34} = 0.104 · 10^{-6} K_{36} = 0.492 · 10^{-6} K_{50} = 0.239 · 10^{-15} K_{52} = 0.213 · 10^{-14} K_{54} = 0.150 · 10^{-13} K_{56} = 0.757 · 10^{-13} K_{58} = 0.320 · 10^{-12} K_{60} = 0.111 · 10^{-11} K_{62} = 0.391 · 10^{-11} K_{64} = 0.917 · 10^{-11} K_{66} = 0.227 · 10^{-10} K_{68} = 0.522 · 10^{-10} K_{70} = 0.112 · 10^{-9} K_{72} = 0.228 · 10^{-9} K_{74} = 0.442 · 10^{-9} K_{76} = 0.816 · 10^{-9}

7°. In the table, the second column gives the nodes, i.e. the roots of $F(x)$, for $2 \leqslant n \leqslant 16$. In the third and fourth columns the weights of the Gaussian and refining quadratures are given. All pertain to the case of integration on $[0,1]$. The nodes and weights, by virtue of their symmetry with respect to 0.5, are given only for the left half of the interval $[0,1]$.

In the last column of the table are given the relative errors $\Gamma_p^{(n)}$ and $K_p^{(n)}$ of the Gaussian and refining quadratures in computing the integral of x^p on $[-1,+1]$ for the first even p for which the quadratures Γ_n and K_n are no longer exact.

All computations were carried out on the machine of the Institute of Theoretical and Experimental Physics. Programs from the library of long-range [[unclear: continuation on next page]] were used.

of A. V. Uskov' s floating-point library and A. Zhivotovsky and V. Pruss' s integer library.

Remark 1. A pair of quadrature formulas—a Gaussian one and a refining one —is convenient to use in solving integral equations with accuracy control in the metric C, since the even nodes of the refining quadrature formula are the nodes of the Gaussian quadrature formula.

Remark 2. A comparison of the corresponding remainder terms $\Gamma_p^{(2n+1)}$ and $K_p^{(n)}$ of the Gaussian and refining quadrature formulas (for example, $\Gamma_{30}^{(15)}$, $\Gamma_{32}^{(15)}$, $\Gamma_{34}^{(15)}$ and $K_{30}^{(7)}$, $K_{32}^{(7)}$, $K_{34}^{(7)}$) shows that, if the required accuracy of integration is not very high, then the refining quadrature formula is only slightly inferior to the Gaussian quadrature formula of the same cost.

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Note: Figure translations are in progress. See original paper for figures.