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Abstract

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ON ESTIMATES OF FUNDAMENTAL MATRICES FOR GENERAL PARABOLIC SYSTEMS WITH CONSTANT COEFFICIENTS

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The present work is devoted to an estimate of the fundamental matrix by means of which the solution of a general boundary-value problem for a homogeneous parabolic system with constant coefficients in a half-space is constructed.

Consider in the half-space E_{n+1}^{+}, whose points have coordinates $x_1, \dots, x_n, t$, with $x_n \geqslant 0$, the problem

$$\mathcal{L}_0\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad B_0\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,t)\bigg|_{x_n=0} = \Phi(x',t), \qquad (1)$$

where $x = (x_1, \dots, x_n)$, $x' = (x_1, \dots, x_{n-1})$, $\mathcal{L}_0$ and B_0 are matrices whose elements are linear differential operators $l_{kj}^0(\partial/\partial x, \partial/\partial t)$ and $B_{qj}^0(\partial/\partial x, \partial/\partial t)$ with complex coefficients. We assume that the system under consideration satisfies the parabolicity condition formulated in (1), and, moreover, contains only highest-order terms, i.e.: 1) there exist integers s_k and t_j $(k, j = 1, \dots, m,\ \max_k s_k = 0)$ and an integer $b > 0$ such that

$$l_{kj}^0(i\xi\lambda, p\lambda^{2b}) = \lambda^{s_k + t_j} l_{kj}^0(i\xi, p),$$

where $i = \sqrt{-1}$ and $\xi = (\xi_1, \dots, \xi_n)$; 2) the roots p_s of the polynomial $L(i\xi, p) = \det \mathcal{L}_0(i\xi, p)$ satisfy the condition

$$\operatorname{Re} p_s \leqslant -\delta\xi^{2b} \qquad (\delta > 0,\ \xi^2 = \xi_1^2 + \dots + \xi_n^2).$$

From condition 1) it follows that the polynomial $L(i\xi, p)$ is homogeneous:

$$L(i\xi\lambda, p\lambda^{2b}) = \lambda^{\sum_{k=1}^{m}(s_k + t_k)} L(i\xi, p),$$

and from condition 2), that $L(0,p) = \gamma p^r,\ \gamma \neq 0$, and

$$\sum_{k=1}^{m}(s_k + t_k) = 2br = 2R, \qquad R > 0.$$

The polynomial $L(i\zeta, i\tau, p)$, where $\zeta = (\zeta_1, \dots, \zeta_{n-1})$, has exactly R roots in τ with positive imaginary part (we denote them by τ_s^+) and R with negative imaginary part, if ζ_j are real and

$$\operatorname{Re} p \geqslant -\delta_1 \zeta^{2b}, \qquad |p|^2 + \zeta^{4b} > 0 \qquad (\delta_1 < \delta). \tag{2}$$

It is also assumed that there exist integers σ_q $(q = 1, \dots, R)$ such that

$$B^0_{qj}(i\xi\lambda, p\lambda^{2b}) = \lambda^{\sigma_q + t_j} B^0_{qj}(i\xi, p)$$

and the following algebraic complementing condition is fulfilled: the rows of the matrix

$$A(i\zeta, i\tau, p) = B_0(i\zeta, i\tau, p)\, \mathcal{L}'_0(i\zeta, i\tau, p) \quad (\mathcal{L}'_0 = L\mathcal{L}_0^{-1})$$

as polynomials in τ are linearly independent modulo the polynomial

$$M^+ = \prod_{s=1}^{k} (\tau - \tau_s^+(\zeta, p))$$

for any real ζ and p satisfying condition (2).

Applying in (1) the Fourier transform with respect to the variables $x_1, \dots, x_{n+1}$ and the Laplace transform with respect to t, it is easy to show that the solution $u = (u_1, \dots, u_m)$ of problem (1) is expressed in terms of the given vector $\Phi = (\Phi_1, \dots, \Phi_R)$ by the formula

$$u_j(x,t) = \sum_{q=1}^{R} \int_{-\infty}^{+\infty} d\tau \int_{E_{n-1}} G_{jq}(x' - y', x_n, t - \tau) \Phi_q(y', \tau)\, dy',$$

where

$$G_{jq}(x,t) \frac{1}{(2\pi)^{n-1} 2\pi i} \int_{E_{n-1}} e^{i(x',\zeta)} d\zeta \int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} \widetilde{G}_{jq}(\zeta, p, x_n)\, dp, \tag{3}$$

where $\sigma > -\delta_1 \zeta^{2b}$, and $\widetilde{G}_{jq}$ is the solution of the following boundary-value problem for a system of ordinary differential equations:

$$\sum_{j=1}^{m} l^0_{kj} \left(i\zeta, \frac{d}{dx_n}, p \right) \widetilde{G}_{jq}(\zeta, p, x_n) = 0 \quad (k = 1, \dots, m); \tag{4}$$

$$\sum_{j=1}^{m} B_{sj}^{0}\left(i\zeta, \frac{d}{dx_n}, p\right) \widetilde{G}_{jq}(\zeta, p, x_n)\Big|_{x_n=0} = \delta_{sq}, \qquad \widetilde{G}_{jq} \xrightarrow[x_n \to +\infty]{} 0. \tag{5}$$

Here $s, q = 1, \ldots, R$; δ_{sq} is the Kronecker symbol.

The question of estimating the functions (3) for systems parabolic in the sense of Petrovskii was studied in works of T. Ya. Zagorskii and S. D. Eidelman. T. Ya. Zagorskii (^{2}) proposed for this purpose a number of techniques connected with deformation of the contours along which the integration in (3) is performed. However, in (^{2}) these estimates were carried out without sufficient justification, since one cannot consider it established in (^{2}) that $\widetilde{G}_{jq}$ is analytic in the variables ζ_j and p under general assumptions. In the notes of S. D. Eidelman (^{3},^{4}) more precise estimates of the functions $\widetilde{G}_{jq}$ were formulated than in (^{2}) (they are extremely sharp). In (^{4}) it is indicated that the analyticity of $\widetilde{G}_{jq}$ can be derived from results of V. P. Mikhailov (^{5}) on the solvability of the first boundary-value problem for Petrovskii-parabolic systems and from an algebraic theorem of Ya. B. Lopatinskii (^{6}). In obtaining estimates of solutions of boundary-value problems for general parabolic systems (^{1}), the analyticity property of the functions $\widetilde{G}_{jq}$ was also used. It was established with the help of a special additional assumption that the system determined by the transposed matrix $\mathcal{L}_0^*$ has at least one boundary-value problem satisfying the same conditions as problem (1). We shall establish the analyticity of the functions $\widetilde{G}_{jq}$ without this assumption.

First of all, we note that problem (4)–(5), when the complementing condition is satisfied, has only one solution. This assertion was proved in (^{7}). Let now A' be the matrix with elements

$$A'_{qj}(\zeta, p, \tau) = \sum_{s=1}^{R} \alpha_s^{(q,j)}(\zeta, p)\tau^{s-1},$$

which are the remainders obtained by dividing the elements of the matrix A by M^+. The complementing condition is equivalent to the following assertion: if $\operatorname{Im}\zeta = 0$ and p satisfies conditions (2), then the rank of the matrix

$$\mathfrak{A} = \begin{pmatrix} \alpha_1^{(1,1)} \ldots \alpha_R^{(1,1)} \alpha_1^{(1,2)} \ldots \alpha_R^{(1,2)} \ldots \alpha_1^{(1,m)} \ldots \alpha_R^{(1,m)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_1^{(R,1)} \ldots \alpha_R^{(R,1)} \alpha_1^{(R,2)} \ldots \alpha_R^{(R,2)} \ldots \alpha_1^{(R,m)} \ldots \alpha_R^{(R,m)} \end{pmatrix}$$

is equal to R.

Arguing as in (^{7}, ^{8}), it is easy to show that

$$\widetilde{G}_{jq}(\xi, p, x_n) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma^+} \frac{L_{kj}(i\xi, i\tau, p)\, N_{kq}(\xi, p, \tau)}{M^+(\xi, p, \tau)} e^{i\tau x_n}\, d\tau. \qquad (6)$$

Here γ^+ is a contour enclosing all roots τ_s^+; L_{kj} is the algebraic complement of the element l_{kj}^0 in $\mathcal{L}_0$, and

$$N_{kq}(\xi, p, \tau) = \sum_{s=1}^{R} \beta_s^{(k,q)}(\xi, p)\, M_{R-s}^+(\xi, p, \tau),$$

where $\beta_s^{(k,q)}$ are the elements of the right inverse matrix $\mathfrak{B}$ to the matrix $\mathfrak{A}$:

$$\mathfrak{B}^* = \begin{pmatrix} \beta_1^{(1,1)} & \dots & \beta_R^{(1,1)} & \beta_1^{(2,1)} & \dots & \beta_R^{(2,1)} & \dots & \beta_1^{(m,1)} & \dots & \beta_R^{(m,1)} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_1^{(1,R)} & \dots & \beta_R^{(1,R)} & \beta_1^{(2,R)} & \dots & \beta_R^{(2,R)} & \dots & \beta_1^{(m,R)} & \dots & \beta_R^{(m,R)} & \end{pmatrix}^*,$$

and M_q^+ are polynomials related to M^+ in the following way: if

$$M^+ = \sum_{s=1}^{R} m_s(\xi, p)\, \tau^{R-s}, \quad \text{then} \quad M_q^+ = \sum_{s=1}^{q} m_s(\xi, p)\, \tau^{q-s} \quad (q = 1, \dots, R).$$

The choice of the matrix $\mathfrak{B}$ may be nonunique, but since $\widetilde{G}_{jq}$ is unique, this nonuniqueness plays no role. We shall establish all the necessary properties of the functions $\widetilde{G}_{qj}$ by choosing as $\mathfrak{B}$ various right inverse matrices.

Let $\Delta_1, \dots, \Delta_N(\xi, p)$ be all non-identically-zero determinants of the square matrices $\mathfrak{A}_i(\xi, p)$ of order R, composed of columns of the matrix $\mathfrak{A}$. By the complementarity condition, for the values of ξ and p under consideration they have no common zeros. They are homogeneous:

$$\Delta_i(\lambda\xi, \lambda^{2b} p) = \lambda^{k_i} \Delta_i(\xi, p) \quad (\lambda > 0).$$

Choose numbers $\chi_i \geq 1$ so that $\Delta_i^{\chi_i}$ have the same degree of homogeneity. Let

$$D(\xi, p) = \sum_{i=1}^{N} \Delta_i^{\chi_i}\, \overline{\Delta}_i^{\chi_i} = \sum_{i=1}^{N} |\Delta_i|^{2\chi_i}, \qquad \Delta^{(i)}(\xi, p) = \frac{\Delta_i^{\chi_i - 1} \overline{\Delta}_i^{\chi_i}}{D}.$$

We have $\sum_{i=1}^{N} \Delta_i \Delta^{(i)} = 1$. Now one may put

$$\mathfrak{B} = \sum_{i=1}^{N} \Delta^{(i)} \mathfrak{B}_i, \tag{7}$$

where $\mathfrak{B}_i$ is the matrix composed of zero rows and rows of the matrix $\widehat{\mathfrak{A}}_i$, reciprocal to $\mathfrak{A}_i$, in such a way that $\mathfrak{A}\mathfrak{B}_i = \Delta_i E$ (E is the identity matrix of order R).

If, as $\beta_s^{(k,q)}$, one takes the elements of the matrix (7), then from (6) it is easy to see that $\widetilde{G}_{jq}$ are smooth single-valued functions of the variables ξ_i and p, analytic in ξ_j in a neighborhood of the real axis for every p satisfying condition (2). Moreover, $\widetilde{G}_{jq}$ satisfy the necessary homogeneity conditions

$$\widetilde{G}_{jq}\left(\lambda\xi, \lambda^{2b}p, \frac{x_n}{\lambda}\right) = \lambda^{-\sigma_q - t_j} \widetilde{G}_{jq}(\xi, p, x_n).$$

It remains to show that $\widetilde{G}_{iq}$ are analytic in p in the domain (2) under consideration for every real ζ. To this end fix an arbitrary p_0 in this domain. Let

$$D'(\zeta, p, p_0) = \sum_{i=1}^{N} \Delta_i^{\varkappa_i}(\zeta, p)\, \overline{\Delta_i}^{\varkappa_i}(\zeta, p_0),$$

$$\Delta'^{(i)}(\zeta, p, p_0) = \frac{\Delta_i^{\varkappa_i - 1}(\zeta, p)\, \overline{\Delta_i}^{\varkappa_i}(\zeta, p_0)}{D'(\zeta, p, p_0)},$$

$$\mathfrak{B}' = \sum_{i=1}^{N} \Delta'^{(i)}(\zeta, p, p_0)\, \mathfrak{B}_i(\zeta, p).$$

Since $D'(\zeta, p_0, p_0) = D(\zeta, p_0) > 0$, we have $D'(\zeta, p, p_0) \neq 0$ in a neighborhood of the point p_0, and for p close to p_0 the elements of the matrix $\mathfrak{B}'$ may be taken as $\beta_\zeta^{(k,q)}$. But now, since $\overline{\Delta_i}$ do not depend on p, it is clear that $\widetilde{G}_{iq}$ are analytic in p in a neighborhood of p_0. Since p_0 is arbitrary, it follows that $\widetilde{G}_{iq}$ is analytic in p throughout the whole domain (2).

Besides G_{iq}, it is useful to consider the functions

$$G_{iq}^{(Q)}(x, t) = \frac{1}{(2\pi)^{n-1} 2\pi i} \int_{E_{n-1}} e^{i(x', \zeta)}\, d\zeta \int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} \widetilde{G}_{iq}(\zeta, p, x_n)\, \frac{dp}{(p + \zeta^{2b})^Q},$$

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* For economy of space we give the formula for the transposed matrix $\mathfrak{B}^*$.

which possess the property

$$\left[\frac{\partial}{\partial t}+(-1)^b\Delta_{x'}^b\right]^Q G_{iq}^{(Q)}=G_{iq}.$$

Application of the above-mentioned methods of estimating the integral (3) makes it possible to show that for $t<0$, $G_{iq}^{(Q)}=0$, while for $t>0$ the inequality holds

$$\left|D_t^\mu D_x^\nu G_{iq}^{(Q)}\right|\le C_1 t^{-\frac{n-1+2b-\sigma_q-t_j+\nu+2b\mu-2bQ}{2b}}\exp\left(-C_2\frac{|x|^{\frac{2b}{2b-1}}}{t^{\frac{1}{2b-1}}}\right),$$

which, in turn, permits one to obtain sharp estimates of the solution of problem (1) in various norms.

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Note: Figure translations are in progress. See original paper for figures.