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Abstract

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THE CAUCHY PROBLEM FOR A DIFFERENTIAL OPERATOR DECOMPOSING INTO WAVE FACTORS

(Presented by Academician L. S. Pontryagin on 2 XI 1963)

The work is devoted to the Cauchy problem for the equation

$$\mathcal{L}u \equiv \prod_{k=1}^{l} \left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_k^2}\Delta \right)^{r_k} u(x,t) = 0, \tag{1}$$

where $a_1 > a_2 > \cdots > a_l > 0$; $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$; $x = (x_1, \dots, x_n)$.

Let $2m$ be the order of equation (1); $m = r_1 + r_2 + \cdots + r_l$. The initial conditions of the problem are as follows:

$$\left. \frac{\partial^s u}{\partial t^s} \right|_{t=0} = f_s(x), \qquad s = 0, 1, \dots, 2m-1. \tag{2}$$

It is well known (see ([1]), p. 298) that the solution of this general problem can be obtained if the solution of the problem with the following initial conditions is known:

$$\left. \frac{\partial^s u}{\partial t^s} \right|_{t=0} = 0, \qquad s = 0, 1, \dots, 2m-2,$$

$$\left. \frac{\partial^{2m-1} u}{\partial t^{2m-1}} \right|_{t=0} = f(x), \qquad s = 2m-1. \tag{3}$$

In the case when $r_1 = r_2 = \cdots = r_l = 1$, the solution of the problem under consideration is given with the aid of the classical formulas of Herglotz-Petrovskii for the fundamental solution of a homogeneous strictly hyperbolic equation (see ([1]), p. 306, or ([2]), pp. 28-33).

However, even in this case, it is of interest to obtain formulas expressing the solution of the problem through spherical means of the initial functions, i.e., formulas analogous to the well-known formulas giving the solution of the Cauchy problem for the wave equation.

We obtain such formulas for equation (1) with arbitrary r_k. These formulas allow us to determine exactly the degree of smoothness of the initial functions that ensures the existence of a classical solution of the problem. With their aid we can also completely resolve the question of the character of the dependence of the solution of the equation at the vertex of the characteristic cone on the values of the initial functions in each of the regions into which the surface of the characteristic cone is divided by the planes of the initial data; that is, we indicate when and which of these regions will be a lacuna or a weak lacuna, and establish its order.

A **weak lacuna of order** p is a region at the base of the characteristic cone such that an arbitrary sufficiently smooth change of the initial functions in it does not affect the value of the derivatives of order p with respect to t of the solution $u(x,t)$ of the Cauchy problem at the vertex of the cone, whereas $\partial^{p-1}u(x,t)/\partial t^{p-1}$ (for $p \geq 1$) does not possess this property. Obviously, an ordinary lacuna will be a weak lacuna of order $p = 0$.

Let $Q(x,r)$ be the spherical mean of the function $f(x)$,

$$Q(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} f(x + r\xi)\, d\omega_\xi,$$

where ω_n is the surface area of the unit sphere in n-dimensional space, $d\omega_\xi$ is the surface-area element of this sphere, and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ is a unit vector.

We shall agree to regard

$$\frac{\partial^{-k}\varphi(t)}{\partial t^{-k}} = \frac{1}{(k-1)!} \int_0^t (t-\tau)^{k-1} \varphi(\tau)\, d\tau, \qquad k > 0;$$

then the solution of the Cauchy problem (1) with conditions (3) can be represented by means of the following formula:

$$u(x,t) = \frac{(-1)^{m-1}\Gamma((n-1)/2)}{(n-2)!} \prod_{i=1}^{l} a_i^{2r_i} \frac{\partial^{n-2m}}{\partial t^{n-2m}} \sum_{k=1}^{l} \sum_{j=0}^{r_k - 1} c_{kj} v_{kj}(x,t), \qquad (4)$$

where:

a) for $j \leq (n-2)/2$

$$v_{kj}(x,t) = \frac{(-1)^j}{j!\,\Gamma((n-1)/2 - j)} \int_0^{t/a_k} Q(x,r)\, r^{2j+1} \left(t^2 - a_k^2 r^2\right)^{(n-3)/2 - j} dr;$$

b) for odd n and $j > (n-3)/2$

$$v_{kj}(x,t) = \frac{(-1)^j}{j!\,2^{j-(n-3)/2} a_k^{2j+2}}\, t^{n-1} \sum_{s=0}^{j-(n-1)/2} \gamma_s t^s \frac{\partial^s Q(x, t/a_k)}{\partial t^s};$$

c) for even n and $j > (n-2)/2$

$$v_{kj}(x,t) = \frac{(-1)^j}{\sqrt{\pi}\, j!\, 2^{j-(n-2)/2}} \frac{\partial}{\partial t} \left[\frac{t^n}{a_k^{2j+2-n}} \sum_{s=0}^{j-n/2} \gamma_s t^s \frac{\partial^s}{\partial t^s} \int_0^{t/a_k} \frac{Q(x,r)\, r^{n-1}\, dr}{t^{n-1}\sqrt{t^2 - a_k^2 r^2}} \right].$$

The constants c_{kj} and γ_s are as follows:

$$c_{kj} = \frac{1}{(r_k - 1 - j)!} \frac{d^{\,r_k - 1 - j}}{da^{\,r_k - 1 - j}} \left[\frac{1}{P_k(a)} \right]_{a = a_k^2}; \qquad P_k(a) = \frac{\prod_{i=1}^{l} (a - a_i^2)^{r_i}}{(a - a_k^2)^{r_k}};$$

$$\gamma_{j-(n-1)/2} = 1; \qquad \gamma_s = C^s_{j-(n-1)/2} (n+1)(n+2) \cdots (2j - 2s), \quad \text{if } s < j - \frac{n-1}{2}$$

for odd n, and

$$\gamma_{j-n/2} = 1; \qquad \gamma_s = C^s_{j-n/2} (n+2)(n+3) \cdots (2j - 2s), \quad \text{if } s < j - \frac{n}{2}$$

for even n.

First we obtain formula (4) for equation (1) in the case when

$$r_1 = r_2 = \cdots = r_l = 1 \qquad (l = m).$$

For this we prove that the solution of problem (1), (3) in this case can be represented in the form*

$$u(x,t) = \frac{\partial^{\,2-2m}}{\partial t^{\,2-2m}} \sum_{k=1}^{m} c_k u_k(x,t), \tag{5}$$

where c_k are constants depending only on $n, m, a_1, \ldots, a_m$, while $u_k(x,t)$ is the solution of the following Cauchy problem for the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_k^2} \Delta \right) u_k(x,t) = 0,$$

$$u_k\Big|_{t=0} = 0, \qquad \frac{\partial u_k}{\partial t}\Big|_{t=0} = f(x),$$

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* The solution of the problem that interests us in the case $r_1 = \cdots = r_l = 1$ can be obtained by transforming the solution represented by means of the formulas of Gerglotz-Petrovskii. However, it seems simpler to us to use only the known formula for the solution of the Cauchy problem for the wave equation.

and, consequently, for $u_k(x,t)$, both for even and for odd n, the formula holds

$$u_k(x,t) = \frac{a_k^2}{(n-2)!!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^{t/a_k} Q(x,r)\, r\,(t^2 - a_k^2 r^2)^{(n-3)/2}\, dr$$

(see ([3]), pp. 368-374, or ([2]), pp. 35-88); hence, and from (5), formula (4) follows in the case under consideration.

Formula (4) in the general case (i.e., when $l \le m$) for odd n is obtained by us from the formula for the strictly hyperbolic case as the result of a limiting passage, when some of the a_k coincide in the limit. In the general case for even n our formula is obtained by the method of descent.

The Cauchy problem with initial conditions of general form (2) is reduced to the Cauchy problem (3) (see ([1]), p. 298), and the solution of this problem (2) is a linear combination of solutions of problems of type (3) and of their derivatives with respect to t. The uniqueness theorem for problem (1), (2) follows in an obvious way from the corresponding theorem for the wave equation. Using formula (4), we obtain the following theorems. Let C_p, as usual, denote the class of functions possessing continuous derivatives up to order p inclusive.

Theorem 1. *A classical solution of problem (1), (2) exists if*

$$f_s(x) \in C_{2m-1-s+q}, \qquad s = 0, 1, \ldots, 2m-1,$$

where

$$q = \max_{1\le k\le l} r_k + n/2, \qquad \text{if } n \text{ is even},$$

$$q = \max_{1\le k\le l} r_k + (n-1)/2, \qquad \text{if } n \text{ is odd},$$

and for odd n this condition cannot be weakened, because, for example, for functions $f_s(x)$ depending only on the radius $|x|$, the spherical means have the same smoothness as the functions themselves.

Let us now consider the characteristic cone with vertex at the point (x, t). Denote by $S(x, r)$ the sphere in the space $x = (x_1, \dots, x_n)$ with center at the point x and radius r.

Theorem 2. *For odd n, the solution of problem (1), (2) depends on the initial functions in the following way:*

a) if $n > 2m$, then the interior of the sphere $S(x, t/a_1)$, lying in the base of the characteristic cone with vertex at the point (x, t), will be a lacuna;

b) if $n < 2m$, then the interior of the sphere $S(x, t/a_1)$ will be a weak lacuna of order $2m - n$;

c) for $l > 1$, the regions lying between the spheres $S(x, t/a_k)$ and $S(x, t/a_{k+1})$, $k = 1, 2, \dots, (l-1)$, in the base of the characteristic cone with vertex at the point (x, t), will be weak lacunas of order $2m - 2$.

For even n, there are no weak lacunas in the base of the characteristic cone. This follows from the fact that the number $(n-3)/2$ is half-integer, and therefore derivatives of any order with respect to t of the solution will contain the integrals entering formula (4).

Let us note that in the case of a non-decomposing strictly hyperbolic equation, as V. A. Borovikov showed (see ([4])), the regions lying in the base of the characteristic cone and homeomorphic to the regions between the spheres are not weak lacunas.

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Note: Figure translations are in progress. See original paper for figures.