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Abstract

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THEORY OF ELASTICITY

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THE GREEN TENSOR FOR AN ANISOTROPIC ELASTIC MEDIUM WITH SOURCES OF INTERNAL STRESSES

(Presented by Academician Yu. N. Rabotnov on 4 II 1964)

The equations of the classical theory of elasticity can be written in the following compact form:

$$\operatorname{div}\sigma = -f, \qquad \operatorname{Rot}\overline{C}\sigma = 0. \tag{1}$$

Here f is the density of body forces; σ is the stress tensor, related to the strain tensor ε by Hooke' s law: $\sigma = C\varepsilon,\ \varepsilon = \overline{C}\sigma$. The operator $\operatorname{Rot} \equiv \operatorname{rot}(\operatorname{rot})'$ (the prime denotes transposition) has, in Cartesian coordinates, the representation

$$p^{\alpha\beta} = \varepsilon^{\varepsilon\alpha\lambda}\varepsilon^{\varepsilon\nu\rho}\partial_\lambda\partial_\nu q_{\mu\rho}, \tag{2}$$

where $p = \operatorname{Rot} q,\ \partial_\lambda = \partial/\partial x^\lambda$. The following identities are known (^{1}):

$$\operatorname{div}\operatorname{Rot} = 0, \qquad \operatorname{Rot}\operatorname{def} = 0. \tag{3}$$

A solution of (1) that vanishes at infinity may be represented in the form

$$\sigma = G * f \quad \text{or} \quad \sigma^{\alpha\beta}(x) = \int G_i^{\alpha\beta}(x - x_0) f^i(x_0)\, dx_0, \tag{4}$$

where $G_i^{\alpha\beta}(x)$ is the Green tensor for stresses. As was shown in (^{2}), in the case of an arbitrary anisotropic medium the construction of G reduces to solving an algebraic equation of degree 6 with coefficients depending on $C^{\alpha\beta\lambda\mu}$. In a number of important cases an explicit expression for G can be obtained.

The corresponding problem for an anisotropic medium with sources of internal stresses (dislocations, interstitial atoms and vacancies, temperature gradient, etc.) was formulated by Kröner (3) in the form $(f = 0)$

$$\operatorname{div} \sigma = 0, \qquad \operatorname{Rot} \overline{C} \sigma = \eta \quad (\operatorname{div} \eta = 0). \tag{5}$$

Here $\eta_{\alpha\beta}$ is the incompatibility tensor, characterizing the density of sources of internal stresses.

The aim of the present work is to construct a solution of (5) in a form analogous to (4). However, in determining the corresponding Green tensor it is necessary to take into account that the right-hand side of (5) satisfies the additional condition $\operatorname{div} \eta = 0$. Therefore, as a preliminary step, we shall consider the problem, also of independent interest, of decomposing the field of a symmetric second-rank tensor A, vanishing at infinity, into a component A_1^0 of strain type and a component A_2^0 of incompatibility type,

$$A = A_1^0 + A_2^0, \qquad \operatorname{Rot} A_1^0 = 0, \qquad \operatorname{div} A_2^0 = 0. \tag{6}$$

The existence and uniqueness of such a decomposition were proved by Kröner (3), who, however, did not give an algorithm for carrying it out. Below a solution of this problem is presented. For what follows it is convenient to formulate it in terms of the projection operators Π^0 and Θ^0:

$$A_1^0 = \Pi^0 A = \pi^0 * A, \qquad A_2^0 = \Theta^0 A = \vartheta^0 * A;$$

$$\Pi^0 \Pi^0 = \Pi^0, \qquad \Theta^0 \Theta^0 = \Theta^0, \qquad \Pi^0 + \Theta^0 = E; \tag{7}$$

$$\operatorname{Rot} \Pi^0 = 0, \qquad \operatorname{div} \Theta^0 = 0.$$

Here E is the identity operator, $\pi^0(x)$ and $\vartheta^0(x)$ are fourth-rank tensors that are the kernels of the corresponding projection operators. For them one can give the explicit expressions

$$\pi^0 = -\frac{1}{8\pi} \det(2\Delta - \operatorname{grad} \operatorname{div}) \operatorname{div} R,$$

$$\vartheta^0 = -\frac{1}{8\pi} \operatorname{Rot} \operatorname{Rot} R; \qquad R^{\alpha_1 \alpha_2}_{\beta_1 \beta_2}(x) = r(x) \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2}, \tag{8}$$

which should be understood in the sense of generalized functions [4].

Incidentally, let us note that this also solves the following problem. Let $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ and $\operatorname{div}\sigma = 0$. Then $\sigma = \operatorname{Rot}\Phi$, where $\Phi^{\alpha\beta}$ is a tensor of stress functions, for the unique determination of which one may impose the condition $\operatorname{div}\Phi = 0$ [1]. From (8) it follows immediately that

$$\Phi = -\frac{r}{8\pi} * \operatorname{Rot}\sigma. \tag{9}$$

We can now give the definition of the Green tensor H for system (5). Assuming that η is locally integrable and decreases sufficiently rapidly at infinity, represent the solution in the form

$$\sigma = H * \eta \quad \text{or} \quad \sigma^{\alpha\beta}(x) = \int H^{\alpha\beta}_{\lambda\mu}(x - x_0)\eta^{\lambda\mu}(x_0)\, dx_0. \tag{10}$$

Obviously, H must satisfy the first equation of system (5). On the right-hand side of the second equation there must stand the kernel of an operator that coincides with the identity on the subspace of tensors of incompatibility type and whose divergence is equal to zero. But precisely Θ^0 has these properties. Thus

$$\operatorname{div} H = 0, \qquad \operatorname{Rot}\bar{C}H = \vartheta^0. \tag{11}$$

Alongside (7), introduce the more general decomposition

$$A_1 = \Pi A, \qquad A_2 = \Theta A;$$

$$\Pi\Pi = \Pi, \qquad \Theta\Theta = \Theta, \qquad \Pi + \Theta = E, \tag{12}$$

$$\operatorname{Rot}\bar{C}\Pi = 0, \qquad \operatorname{div}\Theta = 0,$$

which can be interpreted as a decomposition of the stress tensor into stresses caused by body forces and internal stresses. Decomposition (7) may be regarded as a special case of (12). It is easy to see that the following representations are possible for Π and Θ:

$$\Pi = -G * \operatorname{div}, \qquad \Theta = E + G * \operatorname{div}. \tag{13}$$

It can be shown that the following operator identity holds:

$$\Theta = \Theta C \Theta^0 \bar{C}. \tag{14}$$

Taking into account that $\sigma = \Theta\sigma$, from (13) and (14) we find an explicit expression for H:

$$H(x) = -\frac{1}{8\pi}\left[C \operatorname{Rot} R(x) + G(x) * \operatorname{div} C \operatorname{Rot} R(x)\right], \tag{15}$$

which solves the stated problem.

In conclusion, we note that for an isotropic medium system (5) was solved by Kröner by another method [3].

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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