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**Abstract**

**Full Text**

A. A. ARSEN' EV

**ASYMPTOTIC PROPERTIES OF THE TRACE OF THE SPECTRAL FUNCTION OF A SELF-ADJOINT ELLIPTIC DIFFERENTIAL OPERATOR OF SECOND ORDER**

*(Presented by Academician S. L. Sobolev, March 11, 1964)*

I. In an  $N$ -dimensional Euclidean space consider the differential equation

$$Lu = \frac{1}{2} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) - V(x)u = 0, \quad x = (x_1, x_2, \dots, x_N), \quad N \geq 2. \tag{1}$$

With respect to the coefficients  $a_{ij}(x)$  and  $V(x)$  we make the following assumptions:

1.  $a_{ij}(x) \in C^{(1,\mu)}$ ;  $a_{ij}(x)$  are bounded in the whole space  $R(N)$ .
- 2.

$$\sum_{i,j=1}^N a_{ij}(x) \alpha_i \alpha_j \geq c \sum_{i=1}^N \alpha_i^2;$$

$c > 0$  and does not depend on  $x$ .

3.  $V(x) \geq 0$ , and for any point  $x$  there is a neighborhood  $O(x)$ , a constant  $c > 0$ , and  $\gamma > 0$ , such that for  $x_1 \in O(x)$  and  $x_2 \in O(x)$  the inequality

$$|V(x_1) - V(x_2)| < c|x_1 - x_2|^\gamma$$

is satisfied.

Let  $\theta(x, y, \lambda)$  be the spectral function of the operator (1). Its construction is described in the work of L. Gårding <sup>(1)</sup>.

In the case of a purely discrete spectrum,

$$\theta(x, y, \lambda) = \sum_{\lambda_n \leq \lambda} \psi_n(x) \psi_n(y),$$

where  $\psi_n(x)$  and  $\lambda_n$  are the eigenfunctions and eigenvalues of the problem

$$L\psi_n + \lambda_n\psi_n = 0, \quad \int \psi_n^2(x) dx = 1.$$

It is known that

$$\theta^2(x, y, \lambda) \leq \theta(x, x, \lambda)\theta(y, y, \lambda).$$

In <sup>(2,3)</sup> the asymptotic behavior of the function  $\theta(x, y, \lambda)$  as  $\lambda \rightarrow \infty$  was studied. For the case of the Laplace and Schrödinger equations the function  $\theta(x, y, \lambda)$  was studied in detail by B. M. Levitan <sup>(5,6)</sup>.

In <sup>(3)</sup> it is proved that

$$\theta(x, x, \lambda) \sim \frac{\lambda^{N/2}}{(2\pi)^{N/2}\Gamma(N/2 + 1)\sqrt{\det \|a_{ij}(x)\|}}. \quad (2)$$

The asymptotic formula (2) is uniform in  $x$  for  $x$  belonging to any bounded set; however, it is not uniform in  $x$  for  $x$  belonging to the whole domain of variation  $R(N)$  of the independent variables. Therefore the problem arises of describing in some way the behavior of the function  $\theta(x, x, \lambda)$  as a whole.

II. Let

$$\theta(f, \lambda) = \int_{-\infty}^{\infty} \theta(x, x, \lambda)f(x) dx.$$

If  $f(x) \in L_1$ , then, as is not difficult to show,

$$\theta(f, \lambda) \sim \frac{\lambda^{N/2}}{(2\pi)^{N/2}\Gamma(N/2 + 1)} \int_{V(x) < \lambda} \frac{f(x) dx}{\sqrt{\det \|a_{ij}(x)\|}}.$$

If  $f(x) \notin L_1$ , it may turn out that  $\theta(f, \lambda) = \infty$  for  $\lambda < \infty$ . However, imposing certain restrictions on the asymptotic behavior of the functions  $f(x)$  and  $V(x)$ , one can prove the formula

$$\theta(f, \lambda) \sim \frac{\lambda^{N/2}}{(2\pi)^{N/2}\Gamma(N/2 + 1)} \int_{V(x) < \lambda} f(x) \left(1 - \frac{V(x)}{\lambda}\right)^{N/2} dx.$$

For the Schrödinger equation, when  $f(x) = 1$ , this formula has long been known <sup>(4)</sup>. For  $f(x) = V^\sigma(x)$  it was obtained by B. M. Levitan in connection with the problem of summing Fourier series by Riesz means <sup>(6)</sup>.

III. We formulate the main result of the present paper. Let  $f(x) \geq 0$ . Put, by definition,

$$B(f, \lambda) = \frac{\lambda^{N/2}}{(2\pi)^{N/2} \Gamma(N/2 + 1)} \int_{V(x) < \lambda} \frac{f(x)}{\sqrt{\det \|a_{ij}(x)\|}} \left(1 - \frac{V(x)}{\lambda}\right)^{N/2} dx.$$

**Theorem 1.** 1. Let the function  $B(f, \lambda)$  satisfy the condition of Korenblum's Tauberian theorem <sup>(7)</sup>: there exist  $\lambda_0$  and  $\gamma$  such that for all  $\lambda' > \lambda'' > \lambda_0$  the inequality

$$\frac{B(f, \lambda')}{B(f, \lambda'')} \leq \left(\frac{\lambda'}{\lambda''}\right)^\gamma$$

holds.

2. Suppose that as  $\delta \rightarrow 0$ , uniformly with respect to  $\lambda > \lambda_0$ , the equality

$$\lim_{\delta \rightarrow 0} \frac{\int_{\max_{|u| < \delta} V(x+u) < \lambda} f(x) dx - \int_{\min_{|u| < \delta} V(x+u) < \lambda} f(x) dx}{\int_{V(x) < \lambda} f(x) dx} = 0.$$

3. There exists an  $\alpha \in (0, 4)$  such that for sufficiently large  $|x| \geq R_0$  the inequality

$$V(x) > |x|^\alpha$$

holds.

4. There exist constants  $A > 0$  and  $\varepsilon > 0$  such that for some  $q$

$$1 < q < \frac{1}{1 - 2/N},$$

$$\int f^q(x) e^{-A|x|^{\frac{\alpha-\varepsilon}{2}}} dx < \infty.$$

Then, as  $\lambda \rightarrow \infty$ ,

$$\theta(f, \lambda) \sim B(f, \lambda).$$

IV. For the case of the Schrödinger equation, i.e. if

$$Lu = \frac{1}{2}\Delta u - V(x)u, \quad (3)$$

the result obtained can be somewhat sharpened.

**Theorem 2.** Let  $\theta(x, y, \lambda)$  be the spectral function of equation (3),

$$B(f, \lambda) = \frac{\lambda^{N/2}}{(2\pi)^{N/2}\Gamma(N/2 + 1)} \int_{V(x) < \lambda} f(x) \left(1 - \frac{V(x)}{\lambda}\right)^{N/2} dx.$$

Suppose:

1.  $B(f, \lambda)$  satisfies condition 1 of Theorem 1.
2. Uniformly in  $\lambda$ ,

$$\lim_{\delta \rightarrow 0} \int_{\max_{|u| < \delta} V(x+u) < \lambda} f(x) dx / \int_{V(x) < \lambda} f(x) dx = 1.$$

3. For some  $p > N/2$ ,

$$\lim_{t \rightarrow 0} \int \left[1 - \int f(x - \sqrt{t}u) e^{-tV(x)} dx / \int f(x) e^{-tV(x)} dx\right]^p e^{-(1-\varepsilon)pu^2} du = 0.$$

Then

$$\theta(f, \lambda) \sim B(f, \lambda).$$

The proof of this theorem is carried out by Ray's method, based on exact estimates of the Green's function of the Cauchy problem for the heat-conduction equation

$$\frac{\partial u}{\partial t} = Lu, \quad u(x, 0) = u_0(x)$$

by means of continual integrals. It is rather cumbersome, and we omit it.

V. Let us say a few words concerning conditions 1-4. Condition 1 means, roughly speaking, that the function  $B(f, \lambda)$  must grow no faster than some power and must not have excessively large jumps.

Let

$$\varphi(\lambda) = \int_{V(x) < \lambda} f(x) dx.$$

Conditions 1 and 2 are fulfilled if the following requirements are fulfilled:

1. There exists a constant  $C > 0$  such that for all  $\theta$  from some interval  $\theta \in [1 - \delta, 1 + \delta]$  and  $\lambda > \lambda_0$ ,

$$\lambda\varphi'(\theta\lambda) > C\varphi(\lambda).$$

- 2.

$$\max_{|u| < \delta} V(x + u) = V(x)(1 + \rho(\delta)),$$

$$\min_{|u| < \delta} V(x + u) = V(x)(1 - \rho(\delta)), \quad \rho(\delta) \rightarrow 0, \delta \rightarrow 0,$$

and these relations are fulfilled uniformly in  $x$ .

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