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Abstract

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ON THE CONTINUABILITY OF THE SCALAR PRODUCT OF HECKE SERIES OF TWO QUADRATIC FIELDS

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1. In the work ([1]), E. Hecke introduced characters of magnitudes (Größencharaktere) in fields of algebraic numbers and studied the properties of Z-functions constructed with the aid of these characters. The results obtained were applied by Hecke ([1]), and later by H. Rademacher and I. P. Kubilius ([2,3]), to multidimensional analytic number theory.

In the present note the scalar product of Hecke Z-functions is studied for two imaginary quadratic fields. Namely, let $K_1 = R(\sqrt{-d_1})$, $K_2 = R(\sqrt{-d_2})$ be imaginary quadratic fields (R is the field of rational numbers, $d_1, d_2 > 0$, $d_1 \neq d_2$); $\eta^{w_1 n}$ and $\xi^{w_2 m}$ are Hecke characters of the fields K_1 and K_2 (w_i is the number of units in the field K_i); N_i is the norm of divisors in K_i ($i = 1, 2$); consider the function

$$Z\left(s, \eta^{w_1 n}, \xi^{w_2 m}\right) = \sum_{N_1 \mathfrak{A} = N_2 \mathfrak{B}} \frac{\eta^{w_1 n}(\mathfrak{A}) \xi^{w_2 m}(\mathfrak{B})}{(N_1 \mathfrak{A})^{2s}}; \tag{1}$$

$\mathfrak{A}, \mathfrak{B}$ are integral divisors in the fields K_1 and K_2; s is a complex variable. It is easy to see that for $\operatorname{Re} s > 1/2$ the series on the right-hand side of (1) converges absolutely and, by virtue of the multiplicativity of the characters η and ξ, decomposes into an absolutely convergent Euler product.

In what follows $\gamma, \gamma_1, \gamma_2, ...$ are absolute constants; the constants occurring in O depend only on the discriminants of the fields K_1 and K_2.

Theorem 1. *For $|m| + |n| \neq 0$, the function (1) is regular in the half-plane* $\operatorname{Re} s > 1/2 - \gamma$.

Theorem 2. *Let $\widehat{\mathfrak{A}}$ be a class of divisors of the field K_1, $\widehat{\mathfrak{B}}$ a class of divisors of the field K_2, and let $f_{\widehat{\mathfrak{A}}, \widehat{\mathfrak{B}}}(X)$ be the number of pairs of prime divisors $(\mathfrak{p}, \mathfrak{q})$ under the condition $N_1 \mathfrak{p} = N_2 \mathfrak{q} \leq X$, $\mathfrak{p} \in \widehat{\mathfrak{A}}$, $\mathfrak{q} \in \widehat{\mathfrak{B}}$. Then the number of such pairs of prime divisors $(\mathfrak{p}, \mathfrak{q})$, with $\mathfrak{p} \in \widehat{\mathfrak{A}}$, $\mathfrak{q} \in \widehat{\mathfrak{B}}$, $N_1 \mathfrak{p} = N_2 \mathfrak{q} \leq X$, $\varphi_1 \leq \arg \alpha \leq \varphi_2$, $\tilde{\varphi}_1 \leq \arg \beta \leq \tilde{\varphi}_2$, is equal to*

$$\frac{w_1 w_2}{4\pi^2}(\varphi_2-\varphi_1)(\tilde\varphi_2-\tilde\varphi_1) f_{\widehat{\mathfrak A},\widehat{\mathfrak B}}(X) + O\left(X e^{-\gamma_1\sqrt{\log X}}\right)$$

(α, β are Hecke ideal numbers corresponding to the divisors $\mathfrak p, \mathfrak q$).

From Theorem 2 one easily obtains the asymptotic uniformity of the distribution of pairs of prime divisors $(\mathfrak p, \mathfrak q)$ with equal norms in similarly expanding contours.

For the particular case $K_1 = R(\sqrt{-1})$, $K_2 = R(\sqrt{-3})$, Theorems 1 and 2 were formulated in a note by the author ([6])*.

* The plan of proof of the main lemma given in § 3 of note ([6]) is unsuccessful and encounters great difficulties which I have not managed to overcome. However, all the theorems and lemmas formulated in ([6]) are correct and are special cases of the results of the present note.

2. Theorems 1 and 2 are obtained from the following Lemma 1 by the method described in § 2 of note ([6]).

Lemma 1. *Let $\hat{\mathfrak A}$ and $\hat{\mathfrak B}$ be classes of divisors of the fields K_1 and K_2; let N be the number of pairs of integral divisors $(\mathfrak A, \mathfrak B)$ satisfying the conditions $\mathfrak A \in \hat{\mathfrak A}$, $\mathfrak B \in \hat{\mathfrak B}$, $N_1\mathfrak A = N_2\mathfrak B \le X$, $\varphi \le \arg\alpha \le \varphi + \Delta$, $\tilde\varphi \le \arg\beta \le \tilde\varphi + \tilde\Delta$, where α, β are Hecke ideal numbers corresponding to the divisors $\mathfrak A, \mathfrak B$. Then*

$$N = h(X, \hat{\mathfrak A}, \hat{\mathfrak B})\,\Delta\tilde\Delta + O(X^{1-\gamma_2}). \tag{2}$$

The function $h(X; \hat{\mathfrak A}, \hat{\mathfrak B})$ can be written explicitly.

We give a sketch of the proof of Lemma 1. Following E. Hecke, to each divisor $\mathfrak A$ of the field K_i we associate an "ideal number" α—an algebraic number from some finite extension of K_i. Let $\hat{\mathfrak A}$ be a class of divisors of the field K_i; there corresponds to it a quadratic form $f(x, y)$ with integral coefficients such that for every integral divisor $\mathfrak A \in \hat{\mathfrak A}$ there exist integers x, y satisfying $N_i\mathfrak A = f(x, y)$, and, moreover, if in the variables x', y' the form $f(x, y)$ is equal to ${x'}^2 + {y'}^2$, then $|\alpha|^2 = {x'}^2 + {y'}^2$, $\arg\alpha = \operatorname{arctg} y'/x'$. Thus the problem reduces to the asymptotic counting of the number of integral points (x, y, z, t) under the conditions $f(x, y) = g(z, t) \le X$, $\operatorname{tg}\varphi \le y'/x' \le \operatorname{tg}(\varphi + \Delta)$, $\operatorname{tg}\tilde\varphi \le t'/z' \le \operatorname{tg}(\tilde\varphi + \tilde\Delta)$, where $f(x, y)$ and $g(z, t)$ are positive definite quadratic forms with integral coefficients, reducible in the variables (x', y') and (z', t') to the form ${x'}^2 + {y'}^2$ and ${z'}^2 + {t'}^2$. The equation $f(x, y) = g(z, t)$ is easily reduced to the equation

$$b^2 - ac = Dt^2, \tag{3}$$

where the numbers a, b, and c must satisfy certain congruences depending on the forms f and g, the number D is positive and also depends only on f and g, and Lemma 1 follows from the following lemma on the distribution of integral points on the hyperboloid (3).

Lemma 2. *Denote by H_t the hyperboloid $b^2 - ac = Dt^2$. Let $\mathfrak{M}$ be the set of points $(a, b, c) \in H_t$ for which $|a|, |b|, |c| < M$; let Ω be a region on H_t with piecewise smooth boundary, $\Omega \subset \mathfrak{M}$; denote by N the number of integral points $(a, b, c) \in \Omega$ such that $a \equiv \alpha_1$, $b \equiv \alpha_2$, $c \equiv \alpha_3 \pmod{d}$; let F' be the length of the boundary of the region $\frac{1}{t}\Omega$, and let the curvature of the boundary of $\frac{1}{t}\Omega$ be of order $O(1)$. Then*

$$N = tm_t(\alpha_1, \alpha_2, \alpha_3, d)\, V_{\frac{1}{t}\Omega} + O\left(t^{1-\gamma_3} \left(\frac{M}{t}\right)^{\gamma_4} F'^{\gamma_5}\right), \tag{4}$$

where $V_{\frac{1}{t}\Omega}$ is the volume of the cone with vertex at the origin whose base is the region $\frac{1}{t}\Omega$.

3. Theorems on the distribution of integral points on hyperboloids of the form $b^2 - ac = D$ were studied in the works of Yu. V. Linnik ([4]) in the case $D < 0$ and of B. F. Skubenko ([5]) in the case $D > 0$. However, in the case of arbitrary D, in formulas of type (3) one cannot obtain a remainder term with power saving. In our case this can be done because the free term of the equation $b^2 - ac = Dt^2$ is divisible by a large square t^2. The plan of the proof of Lemma 2 is as follows.

Along with the hyperboloid $H_t : b^2 - ac = Dt^2$, introduce for consideration the hyperboloid $H_0 : b_0^2 - a_0 c_0 = D$; let the number D be fixed and $t \to \infty$. As in the works ([4,5]), to each point $(a, b, c) \in H_t$ we associate the matrix

$$L = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$$

and the form $\varphi(x, y) = ax^2 + 2bxy + cy^2$ of discriminant Dt^2. We proceed analogously with the points $(a_0, b_0, c_0) \in H_0$.

We shall call an integral point $(a, b, c) \in H_t$ **primitive** if $(a, b, t) = 1$. (In what follows (a, b, c), $L = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$, and $\varphi(x, y) = ax^2 + 2bxy + cy^2$ are considered ...

are regarded as a single object.) It can be proved that for every primitive point $L = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} \in H_t$ there exist an integral matrix A and a point $L_0 = \begin{pmatrix} b_0 & -a_0 \\ c_0 & -b_0 \end{pmatrix} \in H_0$ such that $AL_0A^{-1}t = L$ and $\det A = t$. Moreover, by subjecting the matrix A to certain conditions, one can ensure that, under the mapping $A \to AL_0A^{-1}t = L$, different A' s (for fixed L_0) correspond to different points L.

Thus, in order to count the number of primitive points $L \in \Omega$ satisfying the congruences

$$a \equiv \alpha_1, \qquad b \equiv \alpha_2, \qquad c \equiv \alpha_3 \pmod d \tag{*}$$

it suffices to count the number of integral matrices $A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ of determinant t for which $AL_0A^{-1}t \in \Omega$ and the components b_{ij} lie in such progressions that the congruences (*) are fulfilled. For the number of such matrices one can obtain an asymptotic formula with a power saving in the remainder term, generalizing Lemma 15 of [4].

In the proof of Lemma 2, conversations with B. F. Skubenko were of great help to me. I take this opportunity to express my gratitude to him.

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Note: Figure translations are in progress. See original paper for figures.